# Principal 2-bundles and their gauge 2-groups

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## Abstract

In this paper we introduce principal 2-bundles and show how they are classified by non-abelian Čech cohomology. Moreover, we show that their gauge 2-groups can be described by 2-group-valued functors, much like in classical bundle theory. Using this, we show that, under some mild requirements, these gauge 2-groups possess a natural smooth structure. In the last section we provide some explicit examples.

**MSC:** 55R65, 22E65, 81T13

## Introduction

This paper gives a precise description of globally defined geometric objects, which are classified by non-abelian Čech cohomology. The general philosophy is to realise these geometric structures as categorified principal bundles, i.e., principal bundles, where sets are replaced by categories, maps by functors and commuting diagrams by natural equivalences of functors (satisfying canonical coherence conditions). The control on these categorified geometric structures is the amount of natural transformations that one allows to differ from the identity. For instance, allowing only identity transformations in the definition of a categorified group below makes it accessible as a crossed module and thus in terms of ordinary group theory.

The main enrichment that comes from categorification is the existence of "higher morphisms" between morphisms, that are not present in the set-theoretical setup. Two prominent examples amongst are homotopies between continuous maps and bimodule morphisms between bimodules (as morphisms between rings or $C^*$-algebras). These higher morphisms lead to very rich structure, because it allows a more flexible concept of invertible morphisms, namely invertibility "up to higher morphisms". In the two examples mentioned, these are the concepts of homotopy equivalence and Morita equivalence.

The main idea of this paper is to represent non-abelian cohomology classes as semi-strict principal 2-bundles, i.e., smooth 2-spaces with a locally trivial strict action of a strict Lie 2-group. This particular subclass of principal 2-bundles

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is quite easily accessible, while the theory in its full generality is much more involved (cf. [Bar06]). In the first section, we develop the concept of principal 2-bundles from first principles and show very precisely in the second section, how semi-strict principal 2-bundles are classified by non-abelian Čech cohomology:

**Theorem.** *If $\mathcal{G}$ is a strict Lie 2-group and $M$ is a smooth manifold, then semi-strict principal $\mathcal{G}$-2-bundles over $M$ are classified up to Morita equivalence by $\check{H}(M, \mathcal{G})$.* ■

In addition, we provide a geometric way to think of the band of a semi-strict principal 2-bundle. The initial ideas exposed in this sections are not new, the earliest reference we found was [Ded60]. What is in scope is a clear and down to earth development of the idea in order to make the subject easily accessible.

The paper also aims at opening the subject to infinite-dimensional Lie theory, and we completely neglect the gauge theoretic motivation of the theory (cf. [BS07], [SW08b] and [SW08a] for this). The treatment of symmetry groups of principal 2-bundles in the third section is a first example. There we show how to identify the automorphism 2-group of a semi-strict principal 2-bundle with 2-group valued functors and use this to put smooth structures on these automorphism 2-groups. The main result of the third section is the following.

**Theorem.** *Let $\mathcal{P} \to M$ be a semi-strict principal $\mathcal{G}$-2-bundle. Assume that $M$ is compact, that $\mathcal{G}$ is locally exponential, and that the action of $\mathcal{G}$ on $\mathcal{P}$ is principal. Then $\mathcal{C}^{\infty}(\mathcal{P}, \mathcal{G}_{\mathrm{Ad}})^{\mathcal{G}}$ is a locally exponential strict Lie 2-group with strict Lie 2-algebra $\mathcal{C}^{\infty}(\mathcal{P}, L(\mathcal{G})_{\mathrm{ad}})^{\mathcal{G}}$.* ■

In the classical setup of principal bundles, given as smooth manifolds on which Lie groups act locally trivially, this isomorphism of gauge transformations and group valued functions was the dawning of the global formulation of gauge theories in terms of principal bundles, which lead to its fancy developments.

The last section treats examples, in particular bundle gerbes (or groupoid extensions, much like [LGSX09, Sect. 2], [GS08, Sect. 3], [Moe02, Sect. 4]). This section is not exhaustive, it should give an intuitive idea for relating bundle gerbes and principal 2-bundles and give some further examples. In the end, we provided a short appendix with some basic concepts of locally convex Lie theory.

In comparison to many other expositions of this subject the principal bundles that we consider are globally defined objects, considered as smooth 2-spaces, together with a locally trivial smooth action of a 2-group. Most of the other approaches consider more general base spaces than we do by allowing hypercovers, arbitrary surjective submersions or even more general Lie groupoids. This text only treats Čech groupoids as surjective submersions. Some global aspects of constructing higher bundles can be found in [LGSX09], [GS08, Ex. 2.19], [Moe02, Th. 3.1] and [Bre94, Sect. 2.7], part of which served as a motivation for our construction. However, there is no reference known to the author, which treats principal 2-bundles as smooth *2-spaces with a locally trivial action* of a structure 2-group. The objects that come closest to what we have in mind are

bundle gerbes, as treated in [Mur96], [ACJ05], [SW08c]. Another global way for describing principal bundles is in terms of their transport 2-functors (cf. [SW08b] for ordinary principal bundles and [SW08a] for principal 2-bundles), which is closely related to our approach.

Although there frequently exist more systematic approaches to the things we present, we avoid the introduction of more general frameworks (such as internal categories, 2-categories, etc.). This is done to keep the text quickly accessible.

**Notation:** For a small category $\mathcal{C}$, we shall write $\mathcal{C}_0$ and $\mathcal{C}_1$ for its sets of objects and morphisms. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, then $F_0$ and $F_1$ denotes the maps on objects and morphisms. If $F, G : \mathcal{C} \to \mathcal{D}$ are functors and $\alpha_x : F(x) \to G(x)$ is natural, then we also write $\alpha : F \Rightarrow G$ if we want to emphasise thinking of $\alpha$ as a map $\alpha : \mathcal{C}_0 \to \mathcal{D}_1$. Moreover, we write $\Delta_{\mathcal{C}}$ (or shortly $\Delta$ if $\mathcal{C}$ is understood) for the diagonal embedding $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$. Unless stated otherwise, all categories are assumed to be small.

# I   Principal 2-bundles

In this section we introduce principal 2-bundles. For those readers who wonder what the 2 refers to: throughout this paper we are working *in* the 2-category of categories, where objects are given by categories, morphisms by functors between categories and 2-morphisms by natural transformations between functors. It is *not* the case that the things we call 2-something are 2-categories by themselves (just as a set is not a category but an object in the category of sets).
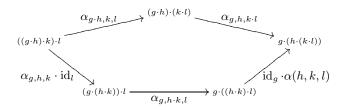
Although the repeated term "...such that there exist natural equivalences..." is quite annoying we state it explicitly every time it occurs, because it is the source of the additional structure in the theory (compared to ordinary bundle theory), which deserves to be pointed out. On the other hand, the occurring coherence conditions in terms of commutative diagrams can safely be neglected at first reading, because they will all be trivially satisfied later on.
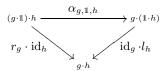
def:2-group

**Definition I.1.** A *weak 2-group* is a monoidal category, in which each morphism is invertible and each object is weakly invertible. We spell this out for convenience. It is given by a category $\mathcal{G}$ together with a multiplication functor $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ (mostly written as $g \cdot h := \otimes(g, h)$), an object $\mathbb{1}$ of $\mathcal{G}$ and natural equivalences

$$\alpha_{g,h,k} : (g \cdot h) \cdot k \to g \cdot (h \cdot k),$$

3

$l_g : \mathbb{1} \cdot g \to g$ and $r_g : g \cdot \mathbb{1} \to g$, such that the diagrams

$$
\begin{array}{ccc}
& ((g\cdot h)\cdot k)\cdot l \xrightarrow{\ \alpha_{g\cdot h,k,l}\ } (g\cdot h)\cdot(k\cdot l) \xrightarrow{\ \alpha_{g,h,k\cdot l}\ } g\cdot(h\cdot(k\cdot l)) & \\
& {\scriptstyle \alpha_{g,h,k}\,\cdot\,\mathrm{id}_l}\searrow \qquad\qquad\qquad\qquad \nearrow{\scriptstyle \mathrm{id}_g\,\cdot\,\alpha(h,k,l)} & \\
& (g\cdot(h\cdot k))\cdot l \xrightarrow{\ \alpha_{g,h\cdot k,l}\ } g\cdot((h\cdot k)\cdot l) &
\end{array}
$$

and

$$
\begin{array}{ccc}
(g\cdot\mathbb{1})\cdot h \xrightarrow{\ \alpha_{g,\mathbb{1},h}\ } g\cdot(\mathbb{1}\cdot h) \\
{\scriptstyle r_g\,\cdot\,\mathrm{id}_h}\searrow \qquad \swarrow{\scriptstyle \mathrm{id}_g\,\cdot\,l_h} \\
g\cdot h
\end{array}
$$

commute. Moreover, we require that each morphism is invertible and that for each object $x$ there exists an object $\overline{x}$ such that $x \cdot \overline{x}$ and $\overline{x} \cdot x$ are isomorphic to $\mathbb{1}$.

A *2-group* is a weak 2-group, together with a *coherent choice* of (weak) inverses, given by an additional functor $\iota : \mathcal{G} \to \mathcal{G}$ and natural equivalences $i_g : g \cdot \iota(g) \to \mathbb{1}$ and $e_g : \iota(g) \cdot g \to \mathbb{1}$, such that

$$
\begin{array}{ccccc}
(g\cdot\iota(g))\cdot g & \xrightarrow{\ i_g\,\cdot\,\mathrm{id}_g\ } & \mathbb{1}\cdot g & \xrightarrow{\ l_g\ } & g \\
{\scriptstyle \alpha_{g,\iota(g),g}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_g} \\
g\cdot(\iota(g)\cdot g) & \xrightarrow{\ \mathrm{id}_g\,\cdot\,e_g\ } & g\cdot\mathbb{1} & \xrightarrow{\ r_g\ } & g
\end{array}
$$

commutes. Morphisms of 2-groups are defined to be weakly monoidal functors of the underlying monoidal category (cf. [BL04]). ∎

Our main reference for 2-groups is [BL04], where our 2-groups are called coherent 2-groups. As also mentioned in [BL04], this is what is also called a (coherent) category with group structure (cf. [Lap83], [Ulb81]).

**Example I.2.** (cf. [BL04, Ex. 34]) Let $\mathcal{C}$ be a category and $\mathcal{A}ut_w(\mathcal{C})$ be the category of equivalences of $\mathcal{C}$. Then $\mathcal{A}ut_w(\mathcal{C})$ is a weak 2-group with respect to composition of functors and natural transformations. There is also a coherent version of this 2-group, cf. [BL04, Ex. 35]. ∎

**Example I.3.** Let $G$ be a group, $A$ be abelian and $f : G \times G \times G \to A$ be a group cocycle, i.e., we have

$$f(gh,k,l) + f(g,h,kl) = f(g,h,k) + f(g,hk,l) + f(h,k,l) \tag{1}$$

`eqn:groupCocycle`

for $g,h,k,l \in G$. Then we define a category $\mathcal{G}_f$ by setting $\mathrm{Ob}(\mathcal{G}_f) := G$ and

$$
\mathrm{Hom}(g,g') = \begin{cases} A & \text{if } g = g' \\ \emptyset & \text{else} \end{cases}
$$

with the composition coming from group multiplication in $A$. Then

$$(g \xrightarrow{a} g) \cdot (h \xrightarrow{b} h) := gh \xrightarrow{ab} gh$$

defines a multiplication functor on $\mathcal{G}_f$ and

$$(g, h, k) \mapsto \left( g \cdot h \cdot k \xrightarrow{f(g,h,k)} g \cdot h \cdot k \right)$$

defines a a natural equivalence. That this natural equivalences make the diagrams from Definition I.1 commutes is equivalent to (1). Thus, $\mathcal{G}_f$ is a weak 2-group and each 2-group is equivalent to such a 2-group (cf. [BL04, Sect. 8.3]).∎

**Definition I.4.** A *smooth 2-space* is a small category $\mathcal{M}$ such that $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_1 {}_s\times_t \mathcal{M}_1$ are smooth manifolds and all structure maps are smooth.

A *smooth functor* $F : \mathcal{M} \to \mathcal{M}'$ between smooth 2-spaces is a functor such that $F_0$ and $F_1$ are smooth maps. Likewise, a *smooth natural transformation* $\alpha : F \Rightarrow G$ between smooth functors is a natural transformation which is smooth as a map $\mathcal{M}_0 \to \mathcal{M}_1$.

Eventually, a *smooth equivalence* between smooth 2-spaces $\mathcal{M}$ and $\mathcal{M}'$ is a smooth functor $F : \mathcal{M} \to \mathcal{M}'$ such that there exist a smooth functor $G : \mathcal{M}' \to \mathcal{M}$ and smooth natural equivalences $G \circ F \Rightarrow \mathrm{id}_{\mathcal{M}}$ and $F \circ G \Rightarrow \mathrm{id}_{\mathcal{M}'}$.∎

In our context, a manifold refers to a Hausdorff space, which is locally homeomorphic to open subsets of a locally convex space such that the coordinate changes are smooth (cf. Appendix A). This definition of a smooth 2-space is a bit more rigid than the concept used frequently in the literature for it requires each space really to be a smooth manifold and not just a smooth (or diffeological) space (cf. [BS07], [BH08]). For our present aim, this concept of a smooth 2-space suffices. Most of the smooth 2-spaces that appear in this article are in fact Lie groupoids, but there is no need to restrict to Lie groupoids a priori.

**Example I.5.** The easiest example of a smooth 2-space is simply a smooth manifold as space of objects with only identity morphisms and the obvious structure maps. ∎

**Definition I.6.** A (strong) *Lie 2-group* is a 2-group which is a smooth 2-space at the same time, such that the functors and natural equivalences occurring in the definition of a 2-group are smooth. ∎
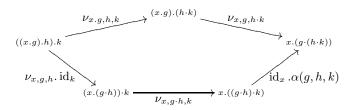
In general, it is quite restrictive to require all functors and natural equivalences to be smooth (cf. [Woc08, Sect. 2] and [Hen08]). However, the major part of this paper deals with strict 2-groups, where the definition is appropriate (cf. [Woc08, Sect. 2]).

We now consider how smooth 2-groups may act on smooth 2-spaces.
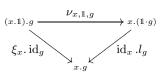
def:G-2-space

**Definition I.7.** Let $\mathcal{G}$ be a Lie 2-group. Then a *smooth $\mathcal{G}$-2-space* is a smooth 2-space $\mathcal{M}$ together with a smooth action, i.e., a smooth functor $\rho : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$
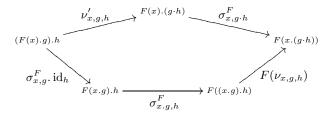
(mostly written as $x.g := \rho(x,g)$) and smooth natural equivalences $\nu : \rho \circ (\rho \times \mathrm{id}_{\mathcal{G}}) \Rightarrow \rho \circ (\mathrm{id}_{\mathcal{M}} \times \otimes)$ and $\xi : \rho \circ (\mathrm{id}_{\mathcal{M}} \times \mathbb{1}) \Rightarrow \mathrm{id}_{\mathcal{M}}$ such that the diagrams

$$
\begin{array}{ccc}
& \overset{\nu_{x.g,h,k}}{\nearrow} (x.g).(h \cdot k) \overset{\nu_{x,g,h\cdot k}}{\searrow} & \\
((x.g).h).k & & x.(g\cdot(h\cdot k)) \\
\nu_{x,g,h}.\,\mathrm{id}_k \searrow & & \nearrow \mathrm{id}_x.\,\alpha(g,h,k) \\
& (x.(g\cdot h))\cdot k \xrightarrow{\nu_{x,g\cdot h,k}} x.((g\cdot h)\cdot k) &
\end{array}
$$

and

$$
\begin{array}{ccc}
(x.\mathbb{1}).g & \xrightarrow{\nu_{x,\mathbb{1},g}} & x.(\mathbb{1}\cdot g) \\
\xi_x.\,\mathrm{id}_g \searrow & & \swarrow \mathrm{id}_x.\,l_g \\
& x.g &
\end{array}
$$

commute. A *morphism* between $\mathcal{G}$-2-spaces is a smooth functor $F : \mathcal{M} \to \mathcal{M}'$, and a smooth natural equivalence $\sigma^F : F \circ \rho \Rightarrow \rho' \circ (F \times \mathrm{id}_{\mathcal{G}})$ such that

$$
\begin{array}{ccc}
& \overset{\nu'_{x,g,h}}{\nearrow} F(x).(g\cdot h) \overset{\sigma^F_{x,g\cdot h}}{\searrow} & \\
(F(x).g).h & & F(x.(g\cdot h)) \\
\sigma^F_{x,g}.\,\mathrm{id}_h \searrow & & \nearrow F(\nu_{x,g,h}) \\
& F(x.g).h \xrightarrow{\sigma^F_{x.g,h}} F((x.g).h) &
\end{array}
$$

commutes. A *2-morphism* between two morphisms $F, F' : \mathcal{M} \to \mathcal{M}'$ of $\mathcal{G}$-2-spaces is a smooth natural transformation $\tau : F \Rightarrow F'$, such that the diagram

$$
\begin{array}{ccc}
F(x).g & \xrightarrow{\tau_x.\,\mathrm{id}_g} & F'(x).g \\
\sigma^F_{x,g} \downarrow & & \downarrow \sigma^{F'}_{x,g} \\
F(x.g) & \xrightarrow{\tau_{x.g}} & F'(x.g)
\end{array}
$$

commutes. An *equivalence* of smooth $\mathcal{G}$-2-spaces is a morphism $F : \mathcal{M} \to \mathcal{M}'$ such that there exists a morphism $F' : \mathcal{M}' \to \mathcal{M}$ and 2-morphisms $F \circ F' \Rightarrow \mathrm{id}_{\mathcal{M}'}$ and $F' \circ F \Rightarrow \mathrm{id}_{\mathcal{M}}$. In this case, $F'$ is called a *weak inverse* of $F$. ∎

Since we shall only consider smooth actions of Lie 2-groups on smooth 2-spaces we suppress this adjective in the sequel. We are now ready to define principal bundles, whose base is a 2-space with only identity morphisms.

`def:principal-2-bundle` **Definition I.8.** Let $\mathcal{G}$ be a Lie 2-group and $M$ be a smooth manifold (viewed as a smooth 2-space with only identity morphisms). A *principal $\mathcal{G}$-2-bundle* over $M$ is a locally trivial $\mathcal{G}$-2-space over $M$. More precisely, it is a smooth $\mathcal{G}$-2-space $\mathcal{P}$, together with a smooth functor $\pi : \mathcal{P} \to M$, such that there exist an open cover $(U_i)_{i \in I}$ of $M$ and equivalences $\Phi_i : \mathcal{P}|_{U_i} \to U_i \times \mathcal{G}$ of $\mathcal{G}$-2-spaces (where

$\mathcal{G}$ acts on $U_i \times \mathcal{G}$ by right multiplication on the second factor). Moreover, we require $\pi|_{U_i} = \mathrm{pr}_1 \circ \Phi_i$ and $\pi \circ \overline{\Phi}_i = \mathrm{pr}_1$ on the nose for a weak inverse $\overline{\Phi}_i$ of $\Phi_i$. Various diagrams, emerging from the natural equivalences, are required to commute to ensure coherence (cf. [Bar06, Sect. 2.5]).

A *morphism* of principal $\mathcal{G}$-2-bundles over $M$ is a morphism $\Phi : \mathcal{P} \to \mathcal{P}'$ of $\mathcal{G}$-2-spaces satisfying $\pi' \circ \Phi = \pi$, and a 2-morphism between two morphisms of principal $\mathcal{G}$-2-bundles is a 2-morphism of the underlying morphisms of strict $\mathcal{G}$-2-spaces. As above, various diagrams are required to commute (cf. [Bar06, Sect. 2.5]). ∎

We suppress an explicit statement of the coherence conditions for brevity. We do not need them in the sequel, as we shall restrict to cases where most natural equivalences are required to be the identity transformation. Note that in the previous sense, a principal $\mathcal{G}$-2-bundle is "locally trivial", i.e., each $\mathcal{P}|_{U_i}$ is equivalent to $U_i \times \mathcal{G}$. In particular, each principal $\mathcal{G}$-2-bundle is a Lie groupoid.

**Lemma I.9.** *Principal $\mathcal{G}$-2-bundles over $M$, together with their morphisms and smooth natural equivalences between morphisms form a 2-category 2-$\mathcal{B}un(M, \mathcal{G})$.*

**Proof.** It is easily checked, that 2-$\mathcal{B}un(M, \mathcal{G})$ actually is a sub-2-category of the 2-category of small categories, functors and natural transformations. ∎

## II  Classification of principal 2-bundles by Čech cohomology

So far, we have clarified the categorification procedure for principal bundles. We now stick to more specific examples of these bundles, which are classified by non-abelian Čech cohomology. The idea is to strictify everything that concerns the action in case of a strict structure group. Strictification means for us to require natural transformations to be the identity.

**Definition II.1.** A *strict 2-group* is a 2-group $\mathcal{G}$, where all natural equivalences between functors occurring in the definition of a 2-group are the identity. A morphism of strict 2-groups is a weak monoidal functor $F : \mathcal{G} \to \mathcal{G}'$ with $F(g \cdot h) = F(G) \cdot F(h)$. ∎

We promised to keep the reader away from 2-categories. However, many formulae and calculations become intuitively understandable in a graphical representation, which we shortly outline in the following remark. The reader who wants to neglect this representation may do so, we shall provide at each stage explicit formulae.

rem:2-groupAs2-category **Remark II.2.** For a diagrammatic interpretation of various formulae and arguments, it is convenient to view a strict 2-group $\mathcal{G}$ not only as a category, but also as a 2-category with one object. The reader unfamiliar with 2-categories
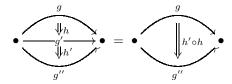
may understand this as the association of an arrow

$$\bullet \xrightarrow{\quad g \quad} \bullet$$

between one fixed object $\bullet$, which we assign to each object $g$ of $\mathcal{G}$, and the association of a 2-arrow



between the arrows $g$ and $g'$, which we assign to each morphism $h : g \to g'$ in $\mathcal{G}$. Then composition in $\mathcal{G}$ is depicted by the vertical composition



of 2-arrows. These diagrams should cause no confusion with the kind of diagrams from the previous section, where objects were represented by points and morphisms were represented by arrows. The latter kind of diagrams will not occur any more in the sequel.

The multiplication functor on objects is then depicted by the horizontal concatenation

$$\bullet \xrightarrow{\quad g \quad} \bullet \xrightarrow{\quad \overline{g} \quad} \bullet \;=\; \bullet \xrightarrow{\quad g \cdot \overline{g} \quad} \bullet$$

of arrows. On morphisms, multiplication is depicted by the horizontal concatenation



of 2-arrows. ∎

We shall only deal with strict 2-groups in the following text, and there are many different ways to describe them. In [BL04, p. 3] one finds the following list (which is explained in detail in [Por08]). A strict 2-group is

- a strict monoidal category in which all objects and morphisms are invertible,

- a strict 2-category with one object in which all 1-morphisms and 2-morphisms are invertible,

- a group object in categories (also called a *categorical group*),

- a category object in groups (also called *internal category* in groups),

- a crossed module.

For the present text, the interpretation of a strict 2-group as a crossed module (and vice versa) will be of central interest, so we recall this concept and relate it to 2-groups.

**Definition II.3.** A *crossed module* consists of two groups $G, H$, an action $\alpha : G \to \mathrm{Aut}(H)$ and a homomorphism $\beta : H \to G$ satisfying
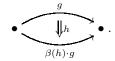
$$\beta(\alpha(g).h) = g \cdot \beta(h) \cdot g^{-1} \tag{2}$$

$$\alpha(\beta(h)).h' = h \cdot h' \cdot h^{-1}. \tag{3}$$

∎

**Remark II.4.** From a crossed module, one can build a 2-group as follows, cf. [Por08], [Bar06, Prop. 16], [FB02] and [Mac98, Sect. XII.8]. The set of objects is $G$ and the set of morphisms is $H \rtimes G$. Each element $(h, g) \in H \rtimes G$ defines a morphism from $g$ to $\beta(h)g$. This is graphically represented by a 2-arrow

$$\bullet \quad \overset{g}{\underset{\beta(h)\cdot g}{\Downarrow h}} \quad \bullet \ .$$

Composition in the category is given by $(h', \beta(h) \cdot g) \circ (h, g) := (h' \cdot h, g)$. This is depicted by defining

$$\bullet \; \overset{g}{\underset{\beta(h')\cdot\beta(h)\cdot g}{\Downarrow h', \; \beta(h)\cdot g, \; \Downarrow h'}} \; \bullet \quad := \quad \bullet \; \overset{g}{\underset{\beta(h'\cdot h)\cdot g}{\Downarrow h'\cdot h}} \; \bullet \ . \tag{4}$$

Consequently, $(e, g)$ defines the identity of $g$. One easily checks that the space of composable pairs of morphisms is $H \rtimes (H \rtimes G)$, where $H \rtimes G$ acts on $H$ by $(h, g).h' = \alpha(\beta(h) \cdot g).h'$ and on this space, composition is given by the homomorphism $(h', (h, g)) \mapsto (h' \cdot h, g)$. Similarly one shows that the space of composable triples of morphisms is $H \rtimes (H \rtimes (H \rtimes G))$ and the associativity in $H$ yields the associativity of composition. The multiplication functor is determined by the group multiplication in $H \rtimes G$. On objects, it is given by multiplication in $G$ and is thus depicted by

$$\bullet \xrightarrow{g} \bullet \xrightarrow{\overline{g}} \bullet \quad := \quad \bullet \xrightarrow{g \cdot \overline{g}} \bullet \quad .$$

On morphisms, it is given by multiplication in $H \rtimes G$ and thus depicted by

$$\bullet \; \overset{g}{\underset{\beta(h)\cdot g}{\Downarrow h}} \; \bullet \; \overset{\overline{g}}{\underset{\beta(\overline{h})\cdot\overline{g}}{\Downarrow \overline{h}}} \; \bullet \quad := \quad \bullet \; \overset{g\cdot\overline{g}}{\underset{\beta(h)\cdot g\cdot\beta(\overline{h})\cdot\overline{g}}{\Downarrow h\cdot(g.\overline{h})}} \; \bullet$$

9

(note that the target of $(h \cdot (g.\overline{h}), g \cdot \overline{g})$ is in fact $\beta(h) \cdot g \cdot \beta(\overline{h}) \cdot \overline{g}$ by (2)). Likewise, the inversion functor is determined by inversion in $H \rtimes G$. All this together defines a strict 2-group.

The reverse construction is also straightforward. For a strict 2-group one checks that objects and morphisms are groups themselves and that all structure maps are group homomorphisms. Then one sets $H$ to be the kernel of the source map and $G$ to be the space of objects. Then $G$ acts on $H$ by $g.h = \mathrm{id}_g \cdot h \cdot \mathrm{id}_{g^{-1}}$ and $\beta$ is given by the restriction of the target map to $H$.

Historically, crossed modules arose first in the work of Whitehead on homotopy 2-types [Whi46], and the equivalence of crossed modules and 2-groups was established by Brown and Spencer in [BS76] and by Loday in [Lod82]. A detailed exposition of the equivalence of the 2-categories of strict 2-groups and of crossed modules is given in [Por08]. ∎

**Remark II.5.** To put more involved diagrams as the ones occurring in the previous remark into formulae, one has to replace

$$\bullet \xrightarrow{\overline{g}} \bullet \Downarrow h \bullet \qquad\qquad \text{by} \qquad\qquad \bullet \Downarrow \overline{g}.h \bullet \qquad\qquad (5)$$

on a $\bullet$ with at least one incoming two outgoing 1-arrows and on a $\bullet$ with at least one outgoing and two incoming 1-arrows one has to replace

$$\bullet \Downarrow h \bullet \xrightarrow{\overline{g}} \bullet \qquad\qquad \text{by} \qquad\qquad \bullet \Downarrow h \bullet. \qquad\qquad (6)$$

This procedure is called *whiskering* and corresponds exactly to multiplication of $(h, g) \in H \rtimes G$ with $(e, \overline{g})$ from the left (in (5)) and from the right (in (6)). By performing these substitutions one ends up in a diagram which depicts compositions of 2-arrows, which can be performed as in (4). That various ways of doing these substitutions yield the same result is encoded in the axioms of a crossed module. We shall see how this works in practise in Definition II.12. ∎

**Definition II.6.** A *smooth crossed module* is a crossed module $(G, H, \alpha, \beta)$ such that $G, H$ are Lie groups, $\beta$ is smooth and the map $G \times H \to H$, $(g, h) \mapsto \alpha(g).h$ is smooth. ∎

**Remark II.7.** Given a smooth crossed module, the pull-back of composable pairs of morphism is $H \rtimes (H \rtimes G)$ and the space of composable triples of morphisms is $H \rtimes (H \rtimes (H \rtimes G))$. From the description in Remark II.4 it follows that all structure maps are smooth and thus the corresponding 2-group actually is a Lie 2-group. ∎

For a strict Lie 2-group to define a smooth crossed module it is necessary that the kernel of the source map is a Lie subgroup (cf. Example II.4). In finite dimensions this is always true but an infinite-dimensional Lie group may possess closed subgroups that are no Lie groups (cf. [Bou89, Ex. III.8.2]). So in the differentiable setup we take smooth crossed modules as the concept from the list on page 8 representing all its equivalent descriptions.

**Definition II.8.** If $\mathcal{G}$ is a strict Lie 2-group, then a $\mathcal{G}$-2-space is called *strict* if all natural equivalences between functors in the definition of a $\mathcal{G}$-2-space are identities. Likewise, a *morphism* between strict $\mathcal{G}$-2-spaces is a morphism $F : \mathcal{M} \to \mathcal{M}'$ of the underlying $\mathcal{G}$-2-spaces with $\sigma^F$ the identity transformation. A *2-morphism* between two morphisms $F, F' : \mathcal{M} \to \mathcal{M}'$ of strict $\mathcal{G}$-2-spaces is 2-morphism $\tau : F \Rightarrow F'$, satisfying $\tau(x.g) = \tau(x).\mathrm{id}_g$. An *equivalence* of strict $\mathcal{G}$-2-spaces is a morphism $F : \mathcal{M} \to \mathcal{M}'$ such that there exist a morphism $F' : \mathcal{M}' \to \mathcal{M}$ and 2-morphisms (all of strict $\mathcal{G}$-2-spaces) $F \circ F' \Rightarrow \mathrm{id}_{\mathcal{M}'}$ and $F' \circ F \Rightarrow \mathrm{id}_{\mathcal{M}}$. In this case, $F'$ is called a *weak inverse* of $F$. ∎

The crucial point for the following text shall be that we restrict to strict $\mathcal{G}$-2-spaces, i.e., we do *not* allow non-identical natural equivalences to occur in any axiom that concerns the action functor $\rho : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$.

**Definition II.9.** If $\mathcal{G}$ is a strict Lie 2-group, then we call a principal $\mathcal{G}$-2-bundle $\mathcal{P}$ *semi-strict* if $\mathcal{G}$ acts strictly on $\mathcal{P}$ and the local trivialisations may be chosen to be equivalences of strict $\mathcal{G}$-2-spaces. A morphism between semi-strict principal $\mathcal{G}$-2-bundles $\mathcal{P}$ and $\mathcal{P}'$ over $M$ (or a *semi-strict bundle morphism* for short) is a morphism $\Phi : \mathcal{P} \to \mathcal{P}'$ of strict $\mathcal{G}$-2-spaces satisfying $\pi' \circ \Phi = \pi$. Likewise, a 2-morphism between two morphisms of semi-strict principal $\mathcal{G}$-2-bundles is a 2-morphism between the underlying morphisms of strict $\mathcal{G}$-2-spaces. ∎

Note that our concept of semi-strictness differs from the one used in [Bar06], which is a normalisation requirement on the cocycles classifying principal 2-bundles. A *strict* principal $\mathcal{G}$-2-bundle would require that all natural equivalences occurring in its definition can be chosen to be the identity. However, this definition is slightly too rigid for a treatment of non-abelian cohomology. On the other hand, many generalisations are possible by increasingly admitting various additional natural equivalences to be non-trivial.

**Lemma II.10.** *Semi-strict principal $\mathcal{G}$-2-bundles over $M$, together with their morphisms and 2-morphisms form a 2-category ss-2-$\mathcal{B}un(M, \mathcal{G})$.*

It shall turn out that semi-strict principal 2-bundles are the immediate generalisations of principal bundles, as long as one is interested in geometric structures defined over ordinary (smooth) spaces, classified in terms of Čech cohomology. Moreover, semi-strict principal 2-bundles are also linked to gerbes as follows.

**Remark II.11.** In general, a gerbe over $X$ is a locally transitive and locally non-empty stack in groupoids (cf. [Moe02]). This means that if $U \mapsto F(U)$ is the underlying fibered category of the stack, then one assumes $X = \bigcup\{U | F(U) \neq \emptyset\}$

and that for objects $a, b$ of $F(U)$, each $x \in U$ has an open neighbourhood $V$ in $U$ with at least one arrow $a|_V \to b|_V$.

Now if $\mathcal{P}$ is a principal $\mathcal{G}$-2-bundle with structure group coming from the smooth crossed module $H \to \mathrm{Aut}(H)$ (and assume that $\mathrm{Aut}(H)$ is a Lie group, for instance if $\pi_0(H)$ is finitely generated, cf. [Bou89]), then this defines a gerbe by

$$U \mapsto \{\text{sections of } \mathcal{P}|_U\}$$

for $U$ an open subset of $M$ (cf. [Moe02, Thm. 3.1]). ∎

From now on we shall assume that $\mathcal{G}$ is a smooth Lie 2-group arising from a smooth crossed module $(\alpha, \beta, H, G)$. We spend the rest of this section on the classification problem for semi-strict principal $\mathcal{G}$-2-bundles over $M$.

It shall turn out that principal $\mathcal{G}$-2-bundles over $M$ are classified (in an appropriate sense) by the non-abelian cohomology $\check{H}(M, \mathcal{G})$. There are many treatments of non-abelian cohomology in the literature, i.e., [Bar06], [BM05], [Bre94], [Gir71], [Ded60], and our definition is essentially the same (with the usual minor conventional differences). Note that we did not put a degree (or dimension) to $\check{H}(M, \mathcal{G})$, for the kind of 2-group that one takes for $\mathcal{G}$ determines its degree in ordinary Čech-cohomology. For instance, $\check{H}(M, \mathcal{G}) = \check{H}^2(M, A)$ if $\mathcal{G}$ is associated to the crossed module $A \to \{*\}$ (for $A$ abelian) and $\check{H}(M, \mathcal{G}) = \check{H}^1(M, G)$ if $\mathcal{G}$ is associated to the crossed module $\{*\} \to G$ (for $G$ arbitrary).

**Definition II.12.** An element in $\check{H}(M, \mathcal{G})$ is represented by an open cover $(U_i)_{i \in I}$ of $M$, together with a collection of smooth maps $g_{ij} : U_{ij} \to G$ and $h_{ijk} : U_{ijk} \to H$ (where multiple lower indices refer to multiple intersections). These maps are required to satisfy point-wise $\beta(h_{ijk}) \cdot g_{ij} \cdot g_{jk} = g_{ik}$ on $U_{ijk}$, i.e.,

 (7)

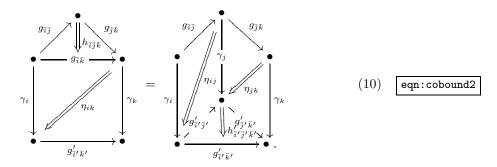and $h_{ikl} \cdot h_{ijk} = h_{ijl} \cdot (g_{ij}.h_{jkl})$ on $U_{ijkl}$, i.e.,

 (8)

(note that the occurrence of $g_{ij}$ in the formula is caused by whiskering, cf. Remark II.5). We furthermore require $g_{ii} = e_G$ and $h_{ijj} = h_{jji} = e_H$ point-wise. We call such a collection $(U_i, g_{ij}, h_{ijk})$ a (non-abelian, normalised, $\mathcal{G}$-valued) *cocycle* on $M$.

Two cocycles $(U_i, g_{ij}, h_{ijk})$ and $(U'_{i'}, g'_{i'j'}, h'_{i'j'k'})$ are called *cohomologous* (or *equivalent*) if there exist a common refinement $(V_i)_{i \in J}$ (i.e., $V_i \subseteq U_{\bar{i}}$ and $V_i \subseteq U'_{\bar{i}'}$

12

for each $i \in J$ and some $\bar{i} \in I$ and $\bar{i}' \in I'$) and a collection of smooth maps $\gamma_i : V_i \to G$ and $\eta_{ij} : V_{ij} \to H$, satisfying point-wise $\gamma_i \cdot g'_{\bar{i}'\bar{j}'} = \beta(\eta_{ij}) \cdot g_{\bar{i}\bar{j}} \cdot \gamma_j$ on $V_{ij}$, i.e.,



(9) `eqn:cobound1`

and $\eta_{ik} \cdot h_{\bar{i}\bar{j}\bar{k}} = \gamma_i . h'_{\bar{i}'\bar{j}'\bar{k}'} \cdot \eta_{ij} \cdot g_{\bar{i}\bar{j}} . \eta_{jk}$ on $V_{ijk}$, i.e.,



(10) `eqn:cobound2`

We furthermore require $\eta_{ii} = e_H$ point-wise. Sometimes, $(V_i, \gamma_i, \eta_{ij})$ is also called a *coboundary* between $(U_i, g_{ij}, h_{ijk})$ and $(U'_{i'}, g'_{i'j'}, h'_{i'j'k'})$. It is easily checked by taking refinements and point-wise products in $\mathcal{G}$ that this defines in fact an equivalence relation and we denote by $\check{H}(M, \mathcal{G})$ the resulting set of equivalence classes of cocycles. ∎

Note that our normalisation conditions $g_{ii} = e_G$, $h_{ijj} = h_{iij} = e_H$ and $\eta_{ii} = e_H$ do *not* imply $g_{ij} = g_{ji}^{-1}$, $h_{ijk} = h_{jik}^{-1}$ and $\eta_{ij} = h_{ji}^{-1}$ as one might expect. Note also that one obtains the non-abelian cohomology as used in the texts, mentioned above, when one takes for a (connected) Lie group $G$ the crossed module $G \to \mathrm{Aut}(G)$, induced by the conjugation homomorphism.

**Remark II.13.** The previous definition is not arbitrary but is the natural generalisation of the following idea. If $G$ is a Lie group, then an ordinary $G$-valued cocycle on $M$ is given by an open cover $(U_i)_{i \in I}$ and smooth $g_{ij} : U_{ij} \to G$ satisfying $g_{ij}g_{jk} = g_{jk}$ and $g_{ii} = e_G$ point-wise. But this is the same as a smooth functor from the Čech groupoid of $(U_i)_{i \in I}$ to the smooth one-object category $BG$. Likewise, coboundaries between cocycles are given by natural transformations between the corresponding functors on refined covers.

If $\mathcal{G}$ is a Lie 2-group, then we can view $\mathcal{G}$ as a 2-category $B\mathcal{G}$ with only one object (cf. Remark II.2). Moreover the Čech groupoid can also be viewed as a 2-category with only identity 2-morphisms. Then the cocyles arise as pseudo (or weak) 2-functors from this 2-category to $B\mathcal{G}$ and coboundaries as pseudonatural transformations between them on refined covers (cf. [Bor94, Sect. 7.5] or [SW08a, App. A] for the terminology). ∎

The point of view from the previous remark also shows that cocycles actually have more categorical structure, i.e., there is a 2-category of cocycles, coming from the 2-category structure of pseudofunctors, pseudonatural transformations and modifications from Čech groupoids to $\mathcal{G}$. However, this lies beyond the scope of the present article and we shall not elaborate on this additional structure.
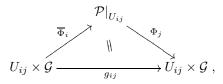
We shall now show how to obtain a cocycle from a principal 2-bundle.

**Remark II.14.** The argument explained below reoccurs frequently in the following construction. For $U \subseteq M$, each morphism of strict $\mathcal{G}$-2-spaces (or strictly equivariant functor) $\Psi : U \times \mathcal{G} \to U \times \mathcal{G}$, which is the identity in the first component, is given by $(x, g) \mapsto (x, g(x)^{-1} \cdot g)$ on objects for a map $g : U \to G$ and $(x, (h, g)) \mapsto (x, (g(x)^{-1}.h, g(x)^{-1} \cdot g))$ on morphisms (it is determined by its values on the subcategory $(U \times \{e_G\}, U \times (e_H, e_G))$ and the artificial inversions is taken in order to match our other conventions, in particular (11)). If $\Psi$ is smooth, then $g$ is smooth and vice versa. For two different such $\Psi_1, \Psi_2$, a 2-morphism $\Psi_1 \Rightarrow \Psi_2$ between morphisms of strict $\mathcal{G}$-2-spaces is given by $(x, g) \mapsto (x, (g_1(x)^{-1}.h(x)^{-1}, g_1(x)^{-1} \cdot g))$ for a unique map $h : U \to H$ satisfying

$$g_2 = (\beta \circ h) \cdot g_1. \tag{11}$$

Moreover, each map $h : U \to H$, satisfying (11), defines a natural equivalence $\Psi_1 \Rightarrow \Psi_2$ which is smooth if and only if $h$ is smooth.

For a principal $\mathcal{G}$-2-bundle $\mathcal{P}$ we now construct a cocycle $z(\mathcal{P})$ as follows. We choose an open cover $(U_i)_{i \in I}$ and local trivialisations $\Phi_i : \mathcal{P}|_{U_i} \to U_i \times \mathcal{G}$, as well as weak inverses $\overline{\Phi}_i : U_i \times \mathcal{G} \to \mathcal{P}|_{U_i}$ and 2-morphisms $\tau_i : \Phi_i \circ \overline{\Phi}_i \Rightarrow \mathrm{id}_{U_i \times \mathcal{G}}$ and $\overline{\tau}_i : \overline{\Phi}_i \circ \Phi_i \Rightarrow \mathrm{id}_{\mathcal{P}|_{U_i}}$. For each pair $i, j \in I$, we consider the composition $\Phi_j \circ \overline{\Phi}_i : U_{ij} \times \mathcal{G} \to U_{ij} \times \mathcal{G}$, of local trivialisations. This is of the above form and thus determined by a smooth map $g_{ij} : U_{ij} \to G$, i.e.,



or, equivalently, $\Phi_j(\overline{\Phi}_i(x, e)) = (x, g_{ij}^{-1}(x))$. Substituting $\Phi_i$ by $(g_{ii}^{-1} \circ \pi) \cdot \Phi_i$, we may assume that $\Phi_i \circ \overline{\Phi}_i = \mathrm{id}_{U_i \times \mathcal{G}}$ on the nose and thus $g_{ii} = e_G$.
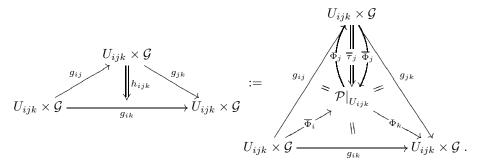
For each $i_1, \ldots, i_n \in I$ with $n \geq 2$ we define the functor

$$\Psi_{i_1 \ldots i_n} := \Phi_{i_n} \circ \overline{\Phi}_{i_{n-1}} \circ \Phi_{i_{n-1}} \circ \cdots \circ \overline{\Phi}_{i_2} \circ \Phi_{i_2} \circ \overline{\Phi}_{i_1}.$$

On objects, $\Psi_{ijk}$ is then given by $(x, g) \mapsto (x, ((g_{ij}(x) \cdot g_{jk}(x))^{-1} \cdot g))$. With $\overline{\tau}_j : \overline{\Phi}_j \circ \Phi_j \Rightarrow \mathrm{id}_{U_j \times \mathcal{G}}$, composition of natural transformations yields a smooth equivalence $\Psi_{ijk} \Rightarrow \Psi_{ik}$ given by a smooth map $h_{ijk} : U_{ijk} \to H$ satisfying

14

$(\beta \circ h_{ijk}) \cdot g_{ij} \cdot g_{jk} = g_{ik}$ by (11), i.e.,



Due to (3), we thus have

$$\Phi_k(\overline{\tau}_j(\overline{\Phi}_i(x,e))) = (g_{ik}(x)^{-1}.h_{ijk}(x)^{-1}, (g_{ij}(x) \cdot g_{jk}(x))^{-1}). \qquad (12)$$
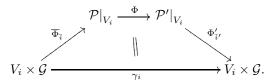
`eqn:fromBundlesToCocycles_`

From $\Phi_i \circ \overline{\Phi}_i = \mathrm{id}_{U_i \times \mathcal{G}}$ it follows that the above natural transformation is the identity if two neighbouring indices agree and thus that $h_{iij} = h_{ijj} = e_H$. Moreover, composition of natural transformations yields smooth equivalences $\Psi_{ijkl} \Rightarrow \Psi_{ijl}$ and $\Psi_{ijkl} \Rightarrow \Psi_{ikl}$, and the compositions $\Psi_{ijkl} \Rightarrow \Psi_{ijl} \Rightarrow \Psi_{il}$ and $\Psi_{ijkl} \Rightarrow \Psi_{ikl} \Rightarrow \Psi_{il}$ coincide (cf. [Bor94, Prop. 1.3.5]). Spelling this out leads to $h_{ikl} \cdot h_{ijk} = h_{ijl} \cdot (g_{ij}.h_{jkl})$. The corresponding diagram can be obtained from plugging the diagram, defining $h_{ijk}$, into (8).

The cocycle $z(\mathcal{P})$ depends on the choices of $(U_i)_{i \in I}$, $\Phi_i$, $\overline{\Phi}_i$ and $\overline{\tau}_i$. However, a special case of the following argument shows that two different such choices lead to cohomologous cocycles.

Assume for the moment that we have fixed some choice of the previous data for each $\mathcal{P}$. Given a morphism $\Phi : \mathcal{P} \to \mathcal{P}'$ of semi-strict principal $\mathcal{G}$-2-bundles over $M$, we shall show that the cocycles $z(\mathcal{P})$ and $z(\mathcal{P}')$, constructed from that data, are cohomologous. First, we choose a common refinement $(V_i)_{i \in J}$ of $(U_i)_{i \in I}$ and $(U'_{i'})_{i' \in I'}$, i.e., $V_i \subseteq U_{\overline{i}}$ and $V_i \subseteq U'_{\overline{i}'}$ for each $i \in J$ and some $\overline{i} \in I$ and $\overline{i}' \in I'$. Clearly, all the choices of $\Phi_{\overline{i}}$'s and $\tau_{\overline{i}}$'s restrict to $V_i$.

For each $i \in J$, we consider the morphism $\Phi'_{\overline{i}'} \circ \Phi \circ \overline{\Phi}_{\overline{i}} : V_i \times \mathcal{G} \to V_i \times \mathcal{G}$. Since $\pi' \circ \Phi = \pi$, this is of the form described initially and thus determined by a smooth map $\gamma_i : V_i \to G$, i.e.,



As above, $\overline{\tau}_j$ and $\overline{\tau}'^{-1}_{\overline{i}}$ give rise to a smooth natural equivalence

$$\underbrace{\Phi'_{\overline{j}'} \circ \Phi \circ \overline{\Phi}_{\overline{j}}}_{=\gamma_j} \circ \underbrace{\Phi_{\overline{j}} \circ \overline{\Phi}_{\overline{i}}}_{=g_{\overline{i}\overline{j}}} \Rightarrow \underbrace{\Phi'_{\overline{j}'} \circ \overline{\Phi}'_{\overline{i}'}}_{=g'_{\overline{i}'\overline{j}'}} \circ \underbrace{\Phi'_{\overline{i}'} \circ \Phi \circ \overline{\Phi}_{\overline{i}}}_{=\gamma_i}$$

15

given by a smooth map $\eta_{ij} : V_{ij} \to H$ satisfying $\beta(\eta_{ij}) \cdot g_{\bar{i}\bar{j}} \cdot \gamma_j = \gamma_i \cdot g'_{\bar{i}'\bar{j}'}$, by (11), i.e.,



Now the compositions

$$\Phi'_{\bar{k}'} \circ \Phi \circ \overline{\Phi}_{\bar{k}} \circ \Phi_{\bar{k}} \circ \overline{\Phi}_{\bar{j}} \circ \Phi_{\bar{j}} \circ \overline{\Phi}_{\bar{i}} \xrightarrow{h_{\bar{i}\bar{j}\bar{k}}} \Phi'_{\bar{k}'} \circ \Phi \circ \overline{\Phi}_{\bar{k}} \circ \Phi_{\bar{k}} \circ \overline{\Phi}_{\bar{i}} \xrightarrow{\eta_{ik}} \Phi'_{\bar{k}'} \circ \overline{\Phi}'_{\bar{i}'} \circ \Phi'_{\bar{i}'} \circ \Phi \circ \overline{\Phi}_{\bar{i}}$$

and

$$\Phi'_{\bar{k}'} \circ \Phi \circ \overline{\Phi}_{\bar{k}} \circ \Phi_{\bar{k}} \circ \overline{\Phi}_{\bar{j}} \circ \Phi_{\bar{j}} \circ \overline{\Phi}_{\bar{i}} \xrightarrow{\eta_{jk}} \Phi'_{\bar{k}'} \circ \overline{\Phi}'_{\bar{j}'} \circ \Phi'_{\bar{j}'} \circ \Phi \circ \overline{\Phi}_{\bar{j}} \circ \Phi_{\bar{j}} \circ \overline{\Phi}_{\bar{i}}$$

$$\xrightarrow{\eta_{ij}} \Phi'_{\bar{k}'} \circ \overline{\Phi}'_{\bar{j}'} \circ \Phi'_{\bar{j}'} \circ \overline{\Phi}'_{\bar{i}'} \circ \Phi'_{\bar{i}'} \circ \Phi \circ \overline{\Phi}_{\bar{i}} \xrightarrow{h'_{\bar{i}'\bar{j}'\bar{k}'}} \Phi'_{\bar{k}'} \circ \overline{\Phi}'_{\bar{i}'} \circ \Phi'_{\bar{i}'} \circ \Phi \circ \overline{\Phi}_{\bar{i}}$$

coincide (cf. [Bor94, Prop. 1.3.5]). Spelling this out leads to

$$\eta_{ik} \cdot h_{\bar{i}\bar{j}\bar{k}} = \gamma_i . h'_{\bar{i}'\bar{j}'\bar{k}'} \cdot \eta_{ij} \cdot g_{\bar{i}\bar{j}} . \eta_{jk}$$

The corresponding diagram can be obtained from plugging the diagrams, defining $h_{ijk}$ and $\eta_{ij}$ into (10).

From the previous argument it also follows that different choices of $(U_i)_{i \in I}$, $\Phi_i$, $\overline{\Phi}_i$, $\tau_i$ and $\overline{\tau}_i$ in the construction of $z(\mathcal{P})$ lead to cohomologous cocycles. In fact, if we apply the construction to the identity of $\mathcal{P}$, then the resulting $\gamma_i : U_i \to G$ and $\eta_{ij} : U_{ij} \to H$ yield the desired coboundary. ∎

prop:fromBundlesToCocycles **Proposition II.15.** *The construction from the previous remark assigns to each semi-strict principal $\mathcal{G}$-2-bundle $\mathcal{P}$ an element $[z(\mathcal{P})] \in \check{H}(M, \mathcal{G})$. Moreover, if for $\mathcal{P}$ and $\mathcal{P}'$ there exists a semi-strict bundle morphism $\Phi : \mathcal{P} \to \mathcal{P}'$, then $[z(\mathcal{P})] = [z(\mathcal{P}')]$.*

What is special about the construction in Remark II.14 is that we can construct a principal 2-bundle $\mathcal{P}_z$ out of a given 2-cocycle $z = (U_i, g_{ij}, h_{ijk})$ with $[z] = [z(\mathcal{P})]$. To illustrate the construction we first recall that for a given group valued cocycle $g_{ij} : U_{ij} \to G$ a corresponding principal bundle is given by

$$P = \bigcup_{i \in I} \{i\} \times U_i \times G/ \sim,$$

where $(i, x, g_{ij}(x) \cdot g) \sim (j, x, g)$. The main idea is to modify this construction by introducing a refined identification. Thinking categorically, this means that we

do not only remember that objects are isomorphic (equivalent), but also track consistently the different isomorphisms that may exist. Identifying isomorphic objects in $\mathcal{P}_z$ then leads to the construction below of the underlying principal $G/\beta(H)$-bundle (cf. Corollary II.25). Moreover, the following construction shall be tailored to generalise the fact that for an ordinary principal bundle, the fibres of the projection map are equivalent to the structure group $G$ as right $G$-spaces.

**Remark II.16.** Given a 2-cocycle $z = (U_i, g_{ij}, h_{ijk})$, we define the category $\mathcal{P}_z$ by

$$\mathrm{Ob}(\mathcal{P}_c) = \bigcup_{i \in I} \{i\} \times U_i \times G$$
$$\mathrm{Mor}(\mathcal{P}_c) = \bigcup_{i,j \in I} \{(i,j)\} \times U_{ij} \times H \times G$$

with the obvious smooth structure. It shall be clear in the sequel that all maps are smooth with respect to this structure, so we shall not comment on this any more. We want the structure maps to make the identification of

$$(i, x, g) \xrightarrow{(i,j,x,h,g)} (j, x, g') \quad \text{with} \quad \begin{array}{c} \bullet \xrightarrow{g} \bullet \\ g_{ij} \searrow \Downarrow h \nearrow g' \\ \bullet \end{array}$$

(the latter diagram is in $\mathcal{G}$) an equivalence of categories. We thus require $g' = g_{ij}^{-1} \cdot \beta(h) \cdot g$. Consequently, the space of composable morphisms is $\coprod U_{ijk} \times H \times G$, which we also endow with the obvious smooth structure. Then composition in $\mathcal{P}_z$ is defined by setting the composition
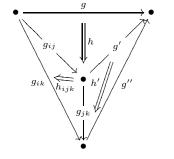
$$(i, x, g) \xrightarrow{(i,j,x,h,g)} (j, x, \underbrace{g_{ij}^{-1} \cdot \beta(h) \cdot g}_{=:g'}) \xrightarrow{(j,k,x,h',g')} (k, x, g_{jk}^{-1} \cdot \beta(h') \cdot g')$$

to be

$$(i, x, g) \xrightarrow{(i,k,x,h_{ijk} \cdot (g_{ij}.h') \cdot h, g)} (k, x, g_{ik}^{-1} \cdot \beta(h_{ijk} \cdot g_{ij}.h' \cdot h) \cdot g),$$

i.e., it is induced by the composition



$$(13)$$

17

in $\mathcal{G}$. That this definition satisfies source-target matching follows from $\beta(h_{ijk}) \cdot g_{ij} \cdot g_{jk} = g_{ik}$. Moreover, $(i, i, x, e, g)$ defines the identity of $(i, x, g)$ and $(i, j, x, h, g)^{-1} = (j, i, x, g_{ij}^{-1}.((h \cdot h_{iji})^{-1}), g_{ij}^{-1} \cdot \beta(h) \cdot g)$. That composition is associative follows from $h_{ikl} \cdot h_{ijk} = h_{ijl} \cdot g_{ij}.h_{jkl}$ (a corresponding equality of diagrams may be obtained from sticking together (13) and (8)). Thus we obtain a smooth 2-space $\mathcal{P}_z$ with an obvious morphism $\mathcal{P}_z \to M$.

The right action of $\mathcal{G}$ on $\mathcal{P}_c$ is given by $(i, x, g).\overline{g} := (i, x, g \cdot \overline{g})$ on objects and by

$$(i, j, x, h, g).(\overline{h}, \overline{g}) := (i, j, x, h \cdot (g.\overline{h}), g \cdot \overline{g})$$

on morphisms, i.e., it is induced by the horizontal composition



We want the local trivialisations $\overline{\Phi}_i$ to be given by the canonical inclusion $U_i \times \mathcal{G} \to \mathcal{P}_z$ and $\Phi_i$ to be tailored such that the natural equivalences $\overline{\tau}_i : \overline{\Phi}_i \circ \Phi_i \Rightarrow \mathrm{id}_{\mathcal{P}_z|_{U_i}}$ are given by

$$(j, x, g) \mapsto \left( (i, x, g_{ij} \cdot g) \xrightarrow{(i,j,x,e,g_{ij} \cdot g)} (j, x, g) \right).$$

We thus set $\Phi_i : \mathcal{P}_z|_{U_i} \to U_i \times \mathcal{G}$ to be induced by the assignment



with $h^* = h_{ijk} \cdot g_{ij}.h$, i.e., $\Phi_i$ is defined by

$$(j, x, g) \mapsto (x, g_{ij} \cdot g)$$
$$(j, k, x, h, g) \mapsto (x, h_{ijk} \cdot g_{ij}.h, g_{ij} \cdot g).$$

In fact, $\Phi_i \circ \overline{\Phi}_i = \mathrm{id}_{U_i \times \mathcal{G}}$ on the nose and $\overline{\tau}_i : \mathrm{id}_{\mathcal{P}_c|_{U_i}} \Rightarrow \overline{\Phi}_i \circ \Phi_i$ is then given by $(j, x, g) \mapsto (i, j, x, e, g_{ij}g)$. ∎
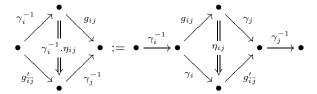
**Proposition II.17.** *For each $\mathcal{G}$-valued cocycle $z$ on $M$, the principal $\mathcal{G}$-2-bundle $\mathcal{P}_z$ over $M$, constructed in the previous remark, has $[z(\mathcal{P}_z)] = [z]$. Moreover, if $z$ is equivalent to $z'$, then there exists a semi-strict principal $\mathcal{G}$-2-bundle $\mathcal{P}$ over $M$, and two semi-strict bundle morphisms $\Phi : \mathcal{P} \to \mathcal{P}_z$ and $\Phi' : \mathcal{P} \to \mathcal{P}_{z'}$.*

**Proof.** Applying the construction of $z(\mathcal{P}_z)$ from Remark II.14 to the bundle $\mathcal{P}_z$, constructed in Remark II.16 (and choosing in this construction $(U_i)_{i \in I}$, $\Phi_i$,
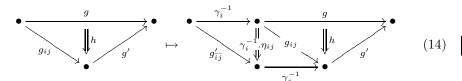
$\overline{\Phi}_i$, $\tau_i$ and $\overline{\tau}_i$ as defined in Remark II.16) shows that with these choices we have in fact $z(\mathcal{P}_z) = z$.

In order to verify the second claim we first consider the case of two cohomologous cocycles $z = (U_i, g_{ij}, h_{ijk})$ and $z' = (U'_{i'}, g'_{i'j'}, h_{i'j'k'})$, where $(U_i)_{i\in I}$ is a refinement of $(U'_{i'})_{i'\in I'}$. I.e., we have $U_i \subseteq U'_{\bar{i}}$ for some $\bar{i} \in I'$, and the coboundary is given by smooth maps $\gamma_i : U_i \to G$ and $\eta_{ij} : U_{ij} \to H$. In this case we set $\mathcal{P}$ to be $\mathcal{P}_z$, $\Phi$ to be $\mathrm{id}_{\mathcal{P}_z}$, and it remains to construct $\Phi' : \mathcal{P}_z \to \mathcal{P}_{z'}$ as follows. First, (9) induces



and this induces $\Phi'$ by the assignment



(14)

that is

$$\Phi'(i, x, g) = (\bar{i}, x, \gamma_i^{-1} \cdot g)$$
$$\Phi'(i, j, x, h, g) = (\bar{i}, \bar{j}, x, \eta_{ij} \cdot (\gamma_i^{-1}.h), \gamma_i^{-1} \cdot g).$$

That $\Phi'$ satisfies source-target matching follows from $\gamma_i \cdot g'_{\bar{i}'\bar{j}'} = \beta(\eta_{ij}) \cdot g_{ij} \cdot \gamma_j$ and that $\Phi'$ preserves identities follows from $\eta_{ii} = e$. That $\Phi'$ commutes with composition follows from $\eta_{ik} \cdot h_{ijk} = \gamma_i.h'_{\bar{i}'\bar{j}'\bar{k}'} \cdot \eta_{ij} \cdot g_{ij}.\eta_{jk}$ (a corresponding equality of diagrams may be obtained from plugging together (14) and (10)). It is obvious from the definition that $\Phi$ commutes strictly with the previously defined action of $\mathcal{G}$. Summarising, $\Phi$ defines a strictly equivariant functor $\mathcal{P}_z \to \mathcal{P}_{z'}$ and, moreover, a morphism of principal 2-bundles.

In the case that the coboundary is given by a common refinement $(V_i)_{i\in I}$, which is properly finer than $(U_i)_{i\in I}$ and $(U'_{i'})_{i'\in I'}$, we proceed as follows. Assume that $V_i \subseteq U_{\bar{i}}$ and that $V_i \subseteq U'_{\bar{i}'}$. Then we restrict $g_{\bar{i}\bar{j}}$ and $g'_{\bar{i}'\bar{j}'}$ to $V_{ij}$, as well as $h_{\bar{i}\bar{j}\bar{k}}$ and $h_{\bar{i}'\bar{j}'\bar{k}'}$ to $V_{\bar{i}\bar{j}\bar{k}}$. This yields refined cocycles $\overline{z}$ and $\overline{z}'$ with the corresponding bundles $\mathcal{P}_{\overline{z}}$, $\mathcal{P}_{\overline{z}'}$ and the canonical inclusion $\Phi : \mathcal{P}_{\overline{z}} \to \mathcal{P}_z$. Since $(V_i)_{i\in I}$ is a refinement of itself, the previous construction yields morphisms $\mathcal{P}_{\overline{z}} \to \mathcal{P}_{\overline{z}'}$ and we set $\Phi'$ to be the composition of this morphism with the inclusion $\mathcal{P}_{\overline{z}'} \to \mathcal{P}_{z'}$. ∎

**Remark II.18.** As the previous proposition suggests, $\check{H}(M, \mathcal{G})$ does not classify bundles in the classical sense, for bundle morphisms between different bundles may not be invertible. For instance, each cover $\mathfrak{U} = (U_i)_{i\in I}$ of $M$ gives

19

rise to a cocycle with values in the trivial 2-group. For two different covers, all these cocycles are cohomologous, but one gets a morphism $\mathcal{P}_{\mathfrak{U}} \to \mathcal{P}_{\mathfrak{U}'}$ (where $\mathcal{P}_{\mathfrak{U}}$, as constructed in Remark II.16, is simply the Čech-groupoid associated to $\mathfrak{U}$) if and only if $\mathfrak{U}$ is a refinement of $\mathfrak{U}'$.

However, if we start with a bundle $\mathcal{P}$, extract a classifying cocycle $z(\mathcal{P})$ as in Proposition II.15 (given by the choice of a cover $(U_i)_{i \in I}$, trivialisations $\Phi_i, \overline{\Phi}_i$ and natural transformations $\overline{\tau}_i$) and reconstruct a bundle $\mathcal{P}_{z(\mathcal{P})}$ as in Proposition II.17, then we always have a morphism $\coprod \overline{\Phi}_i : \mathcal{P}_{z(\mathcal{P})} \to \mathcal{P}$. On objects, this morphism is given by $(i, x, g) \mapsto \overline{\Phi}_i(x, g)$ and on morphisms by the equivariant extension of

$$\left( (i, x, g) \xrightarrow{(i,j,x,e,e)} (j, x, g_{ij}^{-1}) \right) \mapsto \left( \overline{\Phi}_i(x, e) \xrightarrow{\overline{\tau}_j(\overline{\Phi}_i(x,e))^{-1}} \overline{\Phi}_j(x, g_{ij}^{-1}) \right) \quad (15)$$ `eqn:morphismsAndopenCovers`

(with the notation from Remark II.14). That (15) satisfies source-target matching follows from the definition of $g_{ij}$ (in the proof of Proposition II.15), implying

$$\overline{\Phi}_j \circ \Phi_j \circ \overline{\Phi}_i(x, e) = \overline{\Phi}_j(x, g_{ij}^{-1})$$

on objects. That (15) is also compatible with composition follows from the fact that $\coprod \overline{\Phi}_i$ is equivariant and from

$$\Phi_k \left( \overline{\tau}_j \big( \overline{\Phi}_i(x, e) \big)^{-1} \circ \overline{\tau}_k \big( \overline{\Phi}_j(x, g_{ij}^{-1}) \big)^{-1} \right) = \Phi_k \left( \overline{\tau}_k \big( \overline{\Phi}_i(x, e) \big)^{-1} \cdot (h_{ijk}, e) \right),$$

which can be verified directly with the aid of (12). Since (15) is obviously smooth and equivariant by its definition, it defines a morphism $\mathcal{P}_{z(\mathcal{P})} \to \mathcal{P}$. Moreover, it follows from the fact that $\mathcal{G}_1$ acts freely on $\mathcal{P}_1$ that this functor is faithful. ∎

**Proposition II.19.** *The smooth functor $\coprod \overline{\Phi}_i : \mathcal{P}_z \to \mathcal{P}$, defined by (15) is a weak equivalence (or strong Morita equivalence) of the underlying Lie groupoids.*

**Proof.** We have to show that

i) the map

$$\underbrace{\{((i, x, g), f) \in (\mathcal{P}_z)_0 \times \mathcal{P}_1 : \Phi_i(x, g) = s(f)\}}_{:=Q} \xrightarrow{\text{ev}} \mathcal{P}_0, \quad ((i, x, g), f) \mapsto t(f)$$

admits local inverses and that

ii) the diagram

$$\bigcup \{(i, j)\} \times U_{ij} \times H \times G \longrightarrow \mathcal{P}_1$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$(\bigcup \{i\} \times U_i \times G) \times (\bigcup \{j\} \times U_j \times G) \longrightarrow \mathcal{P}_0 \times \mathcal{P}_0$$

is a pull-back.

To show i), we choose a local inverse of the target map in a local trivialisation and transform it to $\mathcal{P}$. In fact, it is easily checked that for $((i,x,g),f) \in Q$ the map

$$\pi^{-1}(U_i)_0 \ni y \mapsto \Big( \underbrace{(i,x(y),\beta(h_f^{-1})g(y))}_{\in(\mathcal{P}_z)_0},$$
$$\underbrace{\overline{\tau}_i(y) \circ \overline{\Phi}_i(x(y),(h_f,\beta(h_f^{-1})g(y))) \circ \overline{\tau}_i(\overline{\Phi}_i(x(y),\beta(h_f^{-1})g(y)))^{-1}}_{\in\mathcal{P}_1} \Big),$$

where $x(y)$ and $g(y)$ denote the components of $\Phi_i(y)$ and $h_f$ is defined by $\Phi_i(f) = (x, h_f, h_f^{-1}g(t(f)))$, defines a local (left) inverse for ev, mapping $t(f)$ to $((i,x,g),f)$.

In order to check ii), we verify the universal property directly. If $(i,x,g)$, $(j,y,k)$ and $f$ are given, such that $f$ is a morphism from $\overline{\Phi}_i(x,g)$ to $\overline{\Phi}_j(y,k)$, then $f$ is a morphism in $\pi^{-1}(U_i)$ and in $\pi^{-1}(U_j)$, for both subcategories are full. Thus $\Phi_i(f)$ is a morphism from $(\Phi_i \circ \overline{\Phi}_i)(x,g) = (x,g)$ to $(\Phi_i \circ \overline{\Phi}_j)(y,k) = (y, g_{ji}(y)^{-1}k)$. From this it follows that $x = y$ and that $\Phi_i(f) = (x,(h_f,g))$ with $\beta(h_f)g = g_{ji}^{-1}(x)k$. Thus $(i,j,x,h_{iji}(x)^{-1}h_f,g)$ is a morphism in $\mathcal{P}_z$, which maps to $(i,x,g)$, $(j,x,k)$ and $f$ under the corresponding maps. This morphism is unique, because $\coprod \overline{\Phi}_i$ is faithful. Clearly, if $(i,x,g)$, $(j,x,k)$ and $f$ depend continuously on some parameter, then this morphism does also. ∎

**Definition II.20.** Two semi-strict principal $\mathcal{G}$-2-bundles $\mathcal{P}$ and $\mathcal{P}'$ over $M$ are said to be *Morita equivalent* if there exists a third such bundle $\overline{\mathcal{P}}$ and a diagram



for semi-strict bundle morphisms $\Phi$ and $\Phi'$. ∎

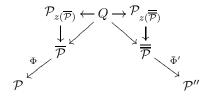**Lemma II.21.** *Morita equivalence of bundles is in fact an equivalence relation.*

**Proof.** Suppose that we are given a diagram



implementing Morita equivalences between $\mathcal{P}$ and $\mathcal{P}'$, and between $\mathcal{P}'$ and $\mathcal{P}''$. Then Proposition II.15 implies that $[z(\overline{\mathcal{P}})] = [z(\overline{\overline{\mathcal{P}}})]$ and thus there exists by Proposition II.17 a bundle $\mathcal{Q}$, together with morphisms $\mathcal{Q} \to \mathcal{P}_{z(\overline{\mathcal{P}})}$ and $\mathcal{Q} \to \mathcal{P}_{z(\overline{\overline{\mathcal{P}}})}$. With the construction from Remark II.18 we can fill in the

morphisms in the diagram

$$\mathcal{P}_{z(\overline{\mathcal{P}})} \leftarrow Q \rightarrow \mathcal{P}_{z(\overline{\overline{\mathcal{P}}})}$$

showing the claim. ∎

With this said, Proposition II.15 and Proposition II.17 now imply the following classification theorem.

**Theorem II.22.** *If $\mathcal{G}$ is a strict Lie 2-group and $M$ is a smooth manifold, then semi-strict principal $\mathcal{G}$-2-bundles over $M$ are classified up to Morita equivalence by $\check{H}(M,\mathcal{G})$.*

**Corollary II.23.** *If $\mathcal{P}$ and $\mathcal{P}'$ are Morita equivalent as principal 2-bundles, then the underlying Lie groupoids are also Morita equivalent.*

We conclude this section with a couple of remarks on the classification result.

**Remark II.24.** Corollary II.23 shows in particular that the Morita equivalence class $[\mathcal{P}]$ of a principal 2-bundle gives rise to a Morita equivalence class of the underlying Lie groupoid and thus determines a smooth stack. Moreover, the Lie 2-group $\mathcal{G}$ determines a group stack $[\mathcal{G}]$. In fact, the Lie groupoid underlying $\mathcal{G}$ can be given the structure of a stacky Lie group [Blo08] by turning the structure morphisms into bibundles as in [Blo08, Sect. 4.6], and this stacky Lie group gives a group stack, cf. [Blo08].

Together with the morphism $[\pi] : [\mathcal{P}] \rightarrow [M]$, the right $[\mathcal{G}]$-action on $[\mathcal{P}]$ and the existence of local trivialisations give rise to something like a principal bundle in the 2-category of smooth stacks. One could have started our investigation with a rigourous definition of this concept and then pursuing a classification of those principal bundles in terms of non-abelian cohomology. This would also have lead to a classification in terms of non-abelian cohomology by very similar arguments. From this point of view it seems natural that non-abelian cohomology can classify principal 2-bundles only up to Morita equivalence. However, our approach is more direct and in more down-to-earth terms. ∎

**Remark II.25.** Let $(G, H, \alpha, \beta)$ be a smooth crossed module such that $\beta(H)$ is a normal split Lie subgroup of $G$ and let $\mathcal{G}$ be the associated Lie 2-group. Then $G/\beta(H)$ carries a natural Lie group structure (cf. [Nee07, Def. 2.1]) and the projection map $G \rightarrow G/\beta(H)$ is smooth. If $\mathcal{P}$ is a semi-strict principal $\mathcal{G}$-2-bundle, then we obtain from this a principal $G/\beta(H)$-bundle $P$ by identifying isomorphic objects in $\mathcal{P}$, i.e., we define $P$ to be $\mathrm{Ob}(\mathcal{P})/\sim$, where $p \sim p'$ if there exists a morphism between $p$ and $p'$. Then $P$ inherits naturally a $G/\beta(H)$-action, given by $[p].[g] := [p.g]$ (where the dot between $p$ and $g$ refers to the $\mathcal{G}$-2-space structure on $\mathcal{P}$) and we endow $P$ with the quotient topology from

22

$\mathrm{Ob}(\mathcal{P})$. Since $\mathcal{P} \to M$ (where $M$ is viewed as a category with only identity morphisms) maps isomorphic objects to the same element in $M$, this functor induces a map $P \to M$ (where $M$ is viewed as a space). If $(U_i)_{i \in I}$ is an open cover such that there exist trivialisations $\Phi_i : \mathcal{P}|_{U_i} \to U_i \times \mathcal{G}$, then $\Phi_i$ induces an $G/\beta(H)$-equivariant bijective map $P|_{U_i} \to U_i \times G/\beta(H)$ and we use this map to endow $P$ with a smooth structure. That this is in fact well-defined follows from the fact that the coordinate changes are then induced by the smooth maps $U_{ij} \to G/\beta(H)$, $x \mapsto [g_{ij}(x)]$, where $g_{ij} : U_{ij} \to G$ is deduced from $\Phi_i$ and $\overline{\Phi}_i$ as in Remark II.14. This turns $P$ into a principal $G/\beta(H)$-bundle, which we call the *band* of $\mathcal{P}$. ∎

**Remark II.26.** Another approach to assign differential geometric data to non-abelian Čech cohomology is to realise classes in $\check{H}(M, \mathcal{G})$ by Morita equivalence classes of Lie groupoid extensions, as outlined in [LGSX09]. In particular, we recover [LGSX09, 3.14] from the above classification by considering the crossed module $H \to G := \mathrm{Aut}(H)$ (for a finite-dimensional $H$ with $\pi_0(H)$ finite, say). For a non-abelian Čech cocycle $z$, Proposition II.17 yields a 2-bundle $\mathcal{P}_z$. Now $G$ acts on the manifolds of objects and morphism of $\mathcal{P}_z$ and since this action is obviously principal and all the structure maps of $\mathcal{P}_z$ are compatible with the $G$-action, we have an induced Lie groupoid $\mathcal{P}_z/G$, with objects $\coprod U_i$ and morphisms $\coprod U_{ij} \times H$. Moreover, the description of the composition in $\mathcal{P}_z$ shows that $\mathcal{P}_z/G$ is exactly the extension of groupoids from [LGSX09, Prop. 3.14].

However, $\mathcal{P}_z/G$ is *not* Morita equivalent to $\mathcal{P}_z$. This can be seen for $M = \{*\}$, where $\mathcal{P}_z$ is the action groupoid of $H$, acting via $\beta$ on $G$ and $\mathcal{P}_z$ is the groupoid with one object and automorphism group $H$. Clearly, $\mathcal{P}_z/G$ is transitive while $\mathcal{P}_z$ is not.

On the other hand, there is an extension of Lie groupoids, canonically associated to each principal 2-bundle, for an arbitrary finite-dimensional crossed module from now on. For this we note that the strong equivalence $\pi^{-1}(U_i) \cong \underline{U_i} \times \mathcal{G}$ yields a weak equivalence [MM03, Prop. 5.11] and we thus have

$$\mathrm{Mor}(\pi^{-1}(U_i)) \cong \{(p, p', (x, (h, g))) \in \mathcal{P}_0 \times \mathcal{P}_0 \times (U_i \times H \rtimes G) :$$
$$\Phi_i(p) = (x, g),\, \Phi_i(p') = (x, \beta(h) \cdot g)\}$$

from the pull-back condition in the definition of weak equivalences. Moreover, the above diffeomorphism is in fact $(H \rtimes G)$-equivariant and we thus see that the action of $\ker(\beta)$ on $\mathrm{Mor}(\pi^{-1}(U_i))$ is principal. We thus have an associated extension

$$
\begin{array}{ccccc}
(\text{identities in } \mathcal{P}_z) \cdot \ker(\beta) & \longrightarrow & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_1/\ker(\beta) \\
\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\mathcal{P}_0 & =\!=\!=\!= & \mathcal{P}_0 & =\!=\!=\!= & \mathcal{P}_0,
\end{array}
\tag{16}
$$

of Lie groupoids.

Note also, that the construction of the band of a Lie groupoid extension from [LGSX09] differs from the construction in Remark II.25, for the band there is a

principal bundle over the space of objects of the considered Lie groupoid, while the band that we construct is a principal bundle over the quotient $\mathcal{P}_0/\mathcal{P}_1$, if it exists as a manifold.

It would be interesting to understand the exact correspondence between our approach and [LGSX09] in more detail. ∎

**Remark II.27.** If $\beta(H)$ is a split Lie subgroup, so that $K := G/\beta(H)$ is again a smooth Lie group, then each non-abelian cocycle determines a smooth $K$-valued 1-cocycle $k_{ij} : U_{ij} \to K$ and because cohomologous cocycles are mapped to cohomologous 1-cocycles, we thus get a map

$$Q : \check{H}(M, \mathcal{G}) \to \check{H}^1(M, K)$$

(realised on bundles by the preceding construction). This map is surjective, for each $U_{ij}$ is contractible, and thus each map $U_{ij} \to K = G/\beta(H)$ has a lift to $G$. The fibres of this map then classify semi-strict principal $\mathcal{G}$-2-bundles with a fixed underlying band. In particular, if $H$ is abelian, the fibre of the trivial band (i.e., all $g_{ij}$ take values in $\beta(H)$) is isomorphic to $H^2(M, H)$. ∎

**Remark II.28.** One can also define a topological version $\check{H}_{\mathrm{top}}(M, \mathcal{G})$ of $\check{H}(M, \mathcal{G})$, where all occurring maps $g_{ij}$, $h_{ijk}$, $\gamma_i$ and $\eta_{ij}$ are required to be continuous rather than smooth. The same classification goes through along the same lines for topological $\mathcal{G}$-2-bundles (for $\mathcal{G}$ a topological 2-group) over an arbitrary topological space $M$. If $M$ is paracompact, then $\check{H}_{\mathrm{top}}(M, \mathcal{G})$ stands in bijection with the set of homotopy classes $[M, B|N\mathcal{G}|]$ and consequently with $\check{H}^1_{\mathrm{top}}(M, |N\mathcal{G}|)$, where $|N\mathcal{G}|$ is a topological group, associated to $\mathcal{G}$ (the geometric realisation of the nerve of the category $\mathcal{G}$). This has been shown in [BS08], cf. also [Jur05] (note that $\mathcal{G}$ is always well-pointed in our case, for we are only dealing with Lie groups). In particular, $\check{H}_{\mathrm{top}}(M, \mathcal{G})$ is trivial if $M$ is paracompact and contractible. This shows that for paracompact finite-dimensional $M$, one can always assume that bundles are trivialised over a fixed good cover and one does not run into the problems described in Remark II.18. A similar approach as in [MW09] should yield the same for $\check{H}(M, \mathcal{G})$. ∎

**Remark II.29.** Assume that $\beta$ is surjective (i.e., assume that $\beta$ is a central extension) and set $\check{\mathcal{H}} := (\{*\}, \ker(\beta))$. Then $H(M, \mathcal{G})$ is isomorphic (as a set) to $\check{H}(M, \mathcal{H})$ for paracompact and finite-dimensional $M$. In fact, if $(g_{ij}, h_{ijk})$ is a non-abelian cocycle, then we define a cohomologous cocycle as follows. First, we assume w.l.o.g. that each $U_{ij}$ is contractible, so that $g_{ij}$ lifts to $\eta_{ij} : U_{ij} \to H$ (assuming $\eta_{ii} \equiv e_H$), and we set $\gamma_i$ to be constantly $e_G$. Then (9) and (10) define a cohomologous cocycle $(g'_{ij}, h'_{ijk})$ and from (7) it follows that $g'_{ij}$ is also constantly $e_G$ and thus $h'_{ijk}$ takes values in $\ker(\beta)$. Thus the canonical map $\check{H}(M, \mathcal{H}) \to H(M, \mathcal{G})$ is surjective and the injectivity follows similarly. Consequently, principal $\mathcal{G}$ 2-bundles are classified (up to Morita equivalence) by $H^2(M, \ker(\beta))$ if $\beta$ is surjective. ∎

**Remark II.30.** There is also a way of understanding the construction in Proposition II.17, given by a construction of 2-bundles as quotients of equivalence 2-relations as in [Bar06, Prop. 22].

Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of $M$. We set $Y := \coprod U_i$ and $\pi_1 : Y \to M$, $(i, x) \mapsto x$. Then $Y^{[n]} := Y \times_M \cdots \times_M Y$ (n-fold fibre product) is the disjoint union of $n$-fold intersections of the $U_i$ and we denote by $\mathcal{U}^{[n]}$ the corresponding category with only identity morphisms. Moreover, we have canonical projections $\pi_{n_1 \ldots n_k} : Y^{[n]} \to Y^{[n-k]}$ for $k < n$, which we identify with the corresponding functors $\pi_{n_1 \ldots n_k} : \mathcal{U}^{[n]} \to \mathcal{U}^{[n-k]}$.

A non-abelian cocycle $c = (g_{ij}, h_{ijk})$, with underlying open cover $\mathcal{U}$, defines what is called a 2-transition in [Bar06, Sect. 2.5.1]. The functor $g : \mathcal{U}^{[2]} \to \mathcal{G}$ (called 2-map in [Bar06]) is given by the smooth map $g : Y^{[2]} \to G$, $((i,j), x) \mapsto g_{ij}(x)$, and the natural isomorphism

$$\gamma : \mu \circ (g \times g) \circ (\pi_{01} \times \pi_{12}) \Rightarrow g \circ \pi_{02}$$

is then given by $Y^{[3]} \ni ((i,j,k), x) \mapsto (h_{ijk}(x), g_{ij}(x) \cdot g_{jk}(x)) \in H \rtimes G$. This 2-transition is semi-strict in the sense of [Bar06] (i.e., the natural $\mu \circ g \circ \iota \Rightarrow \mathbb{1}$ for $\iota : \mathcal{U} \hookrightarrow \mathcal{U}[2]$, $(i, x) \mapsto ((i,i), x)$ is the identity), for $g_{ii} \equiv e_G$ in our setting. Note that $\gamma$ is a natural isomorphism because of condition (7) and the coherence, required in [Bar06, Sect. 2.5.1] is condition (8)).

In [Bar06, Prop. 22], the bundle is constructed from the 2-transition $(g, \gamma)$ by taking the quotient of the category $\mathcal{U}^{[2]} \times \mathcal{G}$ by an equivalence 2-relation, determined by $(g, \gamma)$. This equivalence 2-relation is a categorified version of an equivalence relation, expressed purely in arrow-theoretical terms (cf. [Bar06, 1.1.4 and 2.1.4]).

This equivalence 2-relation is determined by two functors $\mathcal{U}^{[2]} \times \mathcal{G} \to \mathcal{U} \times \mathcal{G}$, one given by $\pi_1 \times \mathrm{id}_\mathcal{G}$ and the other one by

$$(\mathrm{id}_\mathcal{U} \times \rho) \circ (\mathrm{id}_\mathcal{U} \times g \times \mathrm{id}_\mathcal{G}) \circ \iota \times \mathrm{id}_\mathcal{G} .$$

One readily checks that these two functors are what is called jointly 2-monic in [Bar06], for natural equivalences are basically given by $H$-valued mappings, allowing lifts of natural equivalences to be constructed directly. The 2-reflexivity map is given by $\iota \times \mathrm{id}_\mathcal{G}$ (and identities as natural isomorphisms, for our 2-transition is semi-strict). The 2-kernel pair of $\pi_2 \times \mathrm{id}_\mathcal{G}, \pi_1 \times \mathrm{id}_\mathcal{G}$ is simply

$$
\begin{array}{ccc}
\mathcal{U}^{[3]} \times \mathcal{G} & \xrightarrow{\pi_{23} \times \mathrm{id}_\mathcal{G}} & \mathcal{U}^{[2]} \times \mathcal{G} \\
{\scriptstyle \pi_{12} \times \mathrm{id}_\mathcal{G}} \downarrow & & \downarrow {\scriptstyle \pi_1 \times \mathrm{id}_\mathcal{G}} \\
\mathcal{U}^{[2]} \times \mathcal{G} & \xrightarrow{\pi_2 \times \mathrm{id}_\mathcal{G}} & \mathcal{U} \times \mathcal{G}.
\end{array}
$$

The Euclideanness functor is given by $\pi_{13} \times \mathrm{id}_\mathcal{G}$ and the first equivalences in the Euclideanness condition is trivial and the second one is given by $\gamma$ (we choose the 2-kernel pair to be defined by $\pi_1$ and $\pi_0$ so that it fits with the usual notion of an equivalence relation).

It can be checked that the category $\mathcal{P}_c$ is a quotient of this equivalence relation by the inclusion $\mathcal{U} \times \mathcal{G} \hookrightarrow \mathcal{P}_c$ (cf. [Bar06, Sect. 2.1.4]) and thus realises the bundle constructed in [Bar06, Prop. 22]. We leave the details as an exercise. From this construction one sees immediately that the quotient exists in the category of smooth manifolds. ∎

# III Gauge 2-groups

In the classical setup, a gauge transformation of a principal bundle is a bundle self-equivalence and all gauge transformations form a group under composition. Likewise, in the categorified case the vertical self-equivalences form a category (as functors and natural transformations) which is in fact a weak 2-group with respect to the natural compositions.

We will show that this weak 2-group is in fact equivalent to a naturally given strict 2-group. Moreover, we show that under some mild conditions, this strict 2-group carries naturally the structure of a strict *Lie* 2-group. As in the previous section, the fact that we only consider strict actions shall be the crucial point to make the ideas work.

Unless stated otherwise, we assume throughout this section that $\mathcal{G}$ is a strict Lie 2-group arising from the smooth crossed module $(\alpha, \beta, G, H)$ and that $\mathcal{P}$ is a semi-strict principal 2-bundle over the smooth manifold $M$. We will identify $M$ with the smooth 2-space it determines by adding only identity morphisms.

**Remark III.1.** We consider the category $\mathrm{Aut}(\mathcal{P})^{\mathcal{G}}$, whose objects are morphisms $F : \mathcal{P} \to \mathcal{P}$ of principal $\mathcal{G}$-2-bundles and whose morphisms are 2-morphisms $\alpha : F \Rightarrow G$ (cf. Definition I.8). This is a weak 2-group with respect to composition of functors and natural equivalences (cf. [BL04, Ex. 34]). We call this weak 2-group the *gauge 2-group* of $\mathcal{P}$. ∎

We shall make this weak 2-group more accessible by showing that it is equivalent to a strict 2-group. Initially, we start with the most simple case.

**Proposition III.2.** *The category $\mathrm{Aut}(\mathcal{G})^{\mathcal{G}}$ of strictly equivariant endofunctors of $\mathcal{G}$ is equivalent to $\mathcal{G}$.*

**Proof.** Each $F \in \mathrm{Aut}(\mathcal{G})^{\mathcal{G}}$ is given by a functor $F : \mathcal{G} \to \mathcal{G}$ satisfying $F(g \cdot g') = F(g) \cdot g'$ on objects and $F((h, g) \cdot (h', g')) = F((h, g)) \cdot (h', g')$ on morphisms. From this it follows that $F(h, g) = (k_1 \cdot k_2.h, k_2 \cdot g)$, where $F((e, e)) = (k_1, k_2)$, and the compatibility with the structure maps yields $k_1 = e$. Likewise, a natural equivalence between such functors is uniquely given by its value at $e_G$, which is an element of $H$. ∎

Recall that for a category $\mathcal{C}$, we denote by $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ the diagonal embedding and $\Delta_0$ denotes its map on objects.

em:mappingGroupIsA2-Group

**Lemma III.3.** *Let $\mathcal{M}$ be an arbitrary smooth 2-space. Then the category $\mathcal{C}^{\infty}(\mathcal{M}, \mathcal{G})$ of smooth functors from $\mathcal{M}$ to $\mathcal{G}$ is a 2-group with respect to the*

*monoidal functor $\mu_*$, given on objects by $\mu_*(F, G) := \mu \circ (F \times G) \circ \Delta$ and on morphisms by $\mu_*(\alpha, \beta) := \mu_1 \circ (\alpha \times \beta) \circ \Delta_0$. Moreover, if $\mathcal{G}$ is strict, then $\mathcal{C}^\infty(\mathcal{M}, \mathcal{G})$ is so.*

**Proof.** To check that $\mu_*(\alpha, \beta)$ is a natural transformation from $\mu_*(F, F')$ to $\mu_*(G, G')$, one computes that it coincides with the horizontal composition of the natural transformations

$$(\mathrm{id} : \mu \Rightarrow \mu) \circ ((\alpha, \beta) : (F, F') \Rightarrow (G, G')) \circ (\mathrm{id} : \Delta \Rightarrow \Delta).$$

The rest is obvious. ∎

**Remark III.4.** One can easily read off from Lemma III.3 the crossed module $(\alpha_*, \beta_*, G_*, H_*)$, that yields $\mathcal{C}^\infty(\mathcal{M}, \mathcal{G})$ as strict 2-group (cf. Remark II.4). The objects of $\mathcal{C}^\infty(\mathcal{M}, \mathcal{G})$ form a set $\mathcal{M}or(\mathcal{M}, \mathcal{G})$ (as a subset of $C^\infty(\mathcal{M}_0, \mathcal{G}_0) \times C^\infty(\mathcal{M}_1, \mathcal{G}_1)$) and $\mu_*$ defines a group multiplication on this set. Thus we set $G_* = \mathcal{M}or(\mathcal{M}, \mathcal{G})$. Moreover, it is easily checked that $\mathcal{M}or(\mathcal{M}, \mathcal{G})$ is a subgroup of $C^\infty(\mathcal{M}_0, \mathcal{G}_0) \times C^\infty(\mathcal{M}_1, \mathcal{G}_1)$. Likewise, the morphisms in $\mathcal{C}^\infty(\mathcal{M}, \mathcal{G})$ form a set $2\text{-}\mathcal{M}or(\mathcal{M}, \mathcal{G})$ and $\mu_*$ defines a group multiplication on this set. Again, one can interpret $2\text{-}\mathcal{M}or(\mathcal{M}, \mathcal{G})$ as a subgroup

$$2\text{-}\mathcal{M}or(\mathcal{M}, \mathcal{G}) \leq \mathcal{M}or(\mathcal{M}, \mathcal{G}) \times C^\infty(\mathcal{M}_0, \mathcal{G}_1) \times \mathcal{M}or(\mathcal{M}, \mathcal{G}),$$

with $((F, \alpha, G) \in 2\text{-}\mathcal{M}or(\mathcal{M}, \mathcal{G})) :\Leftrightarrow (\alpha : F \Rightarrow G)$. We set $H_*$ to be the kernel of the source map as a subgroup of $C^\infty(\mathcal{M}_0, \mathcal{G}_1) \times \mathcal{M}or(\mathcal{M}, \mathcal{G})$. Then the homomorphism $\beta_* : H_* \to G_*$ is the projection to the second component and the action $\alpha_*$ of $G_*$ on $H_*$ is the conjugation action on the second component.

If $\mathcal{M} = (M, M)$ has only identity morphisms, then $\mathcal{M}or(\mathcal{M}, \mathcal{G}) \cong C^\infty(M, \mathcal{G}_0)$ and $2\text{-}\mathcal{M}or(\mathcal{M}, \mathcal{G}) \cong C^\infty(M, \mathcal{G}_1)$. From this it follows that $\mathcal{C}^\infty(\mathcal{M}, \mathcal{G})$ is associated to the push-forward crossed module $(\alpha_*, \beta_*, C^\infty(M, G), C^\infty(M, H))$, where $\alpha_*$ and $\beta_*$ are the point-wise applications of $\alpha$ and $\beta$. ∎

The following proposition can be understood as an instance of the fact that in the classical case, a bundle endomorphism (covering the identity on the base) of a principal bundle is automatically invertible, and thus bundle endomorphisms form a group. This can best be verified by viewing bundle maps as smooth group-valued maps on the total space. However, note that morphisms between distinct principal bundles need not be invertible (cf. Remark II.18).

**Proposition III.5.** *The weak 2-group $\mathrm{Aut}(\mathcal{P})^\mathcal{G}$ of self-equivalences of $\mathcal{P}$ is equivalent, as a weak 2-group, to $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G}_{\mathrm{Ad}})^\mathcal{G}$, the strict 2-group of morphisms of $\mathcal{G}$-2-spaces, where $\mathcal{G}_{\mathrm{Ad}}$ denotes $\mathcal{G}$ with the conjugation action from the right.*

**Proof.** The existence of strictly equivariant local trivialisations imply that $\mathcal{P}_x := \pi^{-1}(x)$ is equivalent to $\mathcal{G}$. Then the usual reasoning gives $\mathrm{Aut}(\mathcal{P}_x, \mathcal{P}_x)^\mathcal{G} \cong \mathcal{F}un(\mathcal{P}_x, \mathcal{G}_{\mathrm{Ad}})^\mathcal{G}$. Since self-equivalences preserve the subcategories $\mathcal{P}_x$, each object in $\mathrm{Aut}(\mathcal{P})^\mathcal{G}$ is thus given by a strictly equivariant functor $\gamma_F : \mathcal{P} \to \mathcal{G}_{\mathrm{Ad}}$.

That this functor is in fact smooth can be seen in local coordinates. Likewise, each smooth 2-morphism $\alpha : F \Rightarrow G$ between morphisms $F$ and $G$ of $\mathcal{G}$-2-spaces is given by a smooth equivariant map $\eta_\alpha : \mathcal{P}_0 \to H \rtimes G$.

It is readily checked that $F \mapsto \gamma_F$ and $\alpha \mapsto \eta_\alpha$ defines a monoidal functor from $\mathrm{Aut}(\mathcal{P})^{\mathcal{G}}$ to $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G})^{\mathcal{G}}$. The inverse functor from $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G})^{\mathcal{G}}$ to $\mathrm{Aut}(\mathcal{P})^{\mathcal{G}}$ is obviously given by $\gamma \mapsto F_\gamma$ on objects and $\eta \mapsto \alpha_\eta$ on morphisms, where $F_\gamma = \rho \circ (\mathrm{id}_\mathcal{P} \times \gamma) \circ \Delta_\mathcal{P}$ and $\alpha_\eta(p_0) = \mathrm{id}_{p_0} \cdot \eta(p_0)$. ∎

**Remark III.6.** For a semi-strict principal 2-bundle $\mathcal{P}_c$, given by a non-abelian cocycle $c = (h_{ijk}, g_{ij})$, we can also interpret the equivalence $\mathrm{Aut}(\mathcal{P}_c)^{\mathcal{G}} \cong \mathcal{C}^\infty(\mathcal{P}_c, \mathcal{G})^{\mathcal{G}}$ as follows. As we have seen in the proof of Proposition II.15, each self-equivalence of $\mathcal{P}_c$ gives rise to an equivalence of $c$, given by smooth maps $\gamma_i : U_i \to G$ and $\eta_{ij} : U_{ij} \to H$, obeying (9)-(10) and normalisation. This defines a smooth functor $\mathcal{P}_c \to \mathcal{G}$, given on objects by $(i, x, g) \mapsto g \cdot \gamma_i(x)$ and on morphisms by $((i, j), x, (h, g)) \mapsto (\gamma_j(x)^{-1}.(h \cdot \eta_{ij}(x)), g \cdot \gamma_j(x))$. ∎

We now turn to the smoothness conditions on $\mathrm{Aut}(\mathcal{P})^{\mathcal{G}}$. We will endow all spaces of continuous maps with the smooth $C^\infty$ topology, i.e., if $M$ and $N$ are smooth manifolds, then we endow $C^\infty(M, N)$ with the initial topology with respect to

$$C^\infty(M, N) \to \prod_{k=0}^{\infty} C(T^k M, T^k N), \quad f \mapsto (T^k f)_{k \in \mathbb{N}_0}$$

(where $C(T^k M, T^k N)$ is equipped with the compact-open topology). This is the topology on spaces of smooth functions used in [Woc07] and [NW09], whose results we shall cite in the sequel in order to establish Lie 2-group structures on gauge 2-groups.

**Proposition III.7.** *If $M$ is compact, then $\mathcal{C}^\infty(M, \mathcal{G})$ is a Lie 2-group, which is associated to the smooth crossed module $(\alpha_*, \beta_*, C^\infty(M, G), C^\infty(M, H))$.*

**Proof.** We have already seen in Remark III.4 that $\mathcal{C}^\infty(M, \mathcal{G})$ is associated to the push-forward crossed module $(\alpha_*, \beta_*, C^\infty(M, G), C^\infty(M, H))$. This is actually a smooth crossed module, the only non-trivial thing to check is the smoothness of the action of $C^\infty(M, G)$ on $C^\infty(M, H)$. But this follows from the smoothness of parameter-dependent push-forward maps (cf. [Glö02b, Prop. 3.10] and [Woc06, Prop. 28]) and the fact that automorphic actions need only be smooth on unit neighbourhoods in order to be globally smooth. ∎

Before coming to the main result of this section, we have to provide some Lie theory for strict Lie 2-groups by hand.

**Remark III.8.** We briefly recall strict Lie 2-algebras [BC04]. The definition is analogous to that of a strict Lie 2-group as a category in Lie groups. First, a 2-vector space is a category, in which all spaces are vector spaces and all structure maps are linear. A strict Lie 2-algebra is then a 2-vector space $\mathfrak{G}$, together with

a functor $[\cdot, \cdot] : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, which is required to be linear and skew symmetric on objects and morphisms and which satisfies the Jacobi identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

on objects and morphisms.

Coming from strict Lie 2-groups, there is a natural way to associate a strict Lie 2-algebra to a strict Lie 2-group by applying the Lie functor $G \mapsto T_e(G)$, $f \mapsto Tf(e)$. This works, because this functor preserves pull-backs and thus all categorical structures (cf. [BC04, Prop. 5.6]). If $\mathcal{G}$ is a Lie 2-group, then we denote by $L(\mathcal{G})$ the strict Lie 2-algebra one obtains in this way.

We have the same interplay between crossed modules of Lie algebras and strict Lie 2-algebras as in the case of strict Lie 2-groups. A *crossed module* (of Lie algebras) consists of two Lie algebras $\mathfrak{g}, \mathfrak{h}$, and action $\dot{\alpha} : \mathfrak{g} \to \mathrm{der}(\mathfrak{h})$ and a homomorphism $\dot{\beta} : \mathfrak{h} \to \mathfrak{g}$ satisfying $\dot{\beta}(\dot{\alpha}(x).y) = [x, \dot{\beta}(y)]$ and $\dot{\alpha}(\dot{\beta}(x)).y = [x, y]$. To such a crossed module one can associate the Lie 2-algebra $(\mathfrak{g}, \mathfrak{h} \rtimes \mathfrak{g})$ with $s(x, y) = y$, $t(x, y) = \dot{\beta}(x) + y$, $(z, \dot{\beta}(x) + y) \circ (x, y) = (z + x, y)$ and $[\cdot, \cdot]$ given by the Lie-bracket on $\mathfrak{g}$ and $\mathfrak{h} \rtimes \mathfrak{g}$. Moreover, one checks readily that if $\mathcal{G}$ is associated to $(\alpha, \beta, G, H)$, then $L(\mathcal{G})$ is associated to the derived crossed module $(\dot{\alpha}, \dot{\beta}, \mathfrak{h}, \mathfrak{g})$. ∎

**Theorem III.9.** *If $\mathfrak{G}$ is a strict Lie 2-algebra with finite-dimensional object- and morphism space, then there exists a strict Lie 2-group $\mathcal{G}$ such that $L(\mathcal{G})$ is isomorphic to $\mathfrak{G}$.*

**Proof.** There is a functor from Lie algebras to simply connected Lie groups, which is adjoint to the Lie functor. This functor also preserves pull-backs and applied to the spaces of objects and morphisms and to the structure maps of a strict Lie 2-algebra produces a strict Lie 2-group. ∎

**Remark III.10.** If $\mathcal{G}$ is a strict Lie 2-group with strict Lie 2-algebra $\mathfrak{G}$, then we also have a strict 2-action $\mathcal{A}d : \mathfrak{G} \times \mathcal{G} \to \mathfrak{G}$ of $\mathcal{G}$ on $\mathfrak{G}$. This is given on objects and morphisms by $(x, g) \mapsto \mathrm{Ad}(g^{-1}).x$, where Ad is the ordinary adjoint action. That this defines a functor follows from $\mathrm{Ad}(\varphi(g)).\dot{\varphi}(x) = \dot{\varphi}(\mathrm{Ad}(g).x)$ for each homomorphism $\varphi$ of Lie groups. We denote the corresponding $\mathcal{G}$-2-spaces by $\mathcal{G}_{\mathrm{Ad}}$ and $\mathfrak{G}_{\mathrm{ad}}$. ∎

The Lie 2-algebra which is of particular interest in this section is the following.

**Proposition III.11.** *If $\mathcal{M}$ is a strict $\mathcal{G}$-2-space, then $\mathcal{C}^\infty(\mathcal{M}, L(\mathcal{G})_{\mathrm{ad}})^{\mathcal{G}}$, the category of morphisms of $\mathcal{G}$-2-spaces from $\mathcal{M}$ to $L(\mathcal{G})_{\mathrm{ad}}$ is a strict Lie 2-algebra. The functor $[\cdot, \cdot]$ is given by the point-wise application of the functor in $L(\mathcal{G})$ as in Lemma III.3.*

**Proof.** We set $\mathfrak{G} := L(\mathcal{G})$, $\mathfrak{K} := \mathcal{C}^\infty(\mathcal{M}, \mathfrak{G})^{\mathcal{G}}$ and identify $\mathfrak{K}_0$ with a subset of the locally convex Lie algebra $\mathfrak{l} := C^\infty(\mathcal{M}_0, \mathfrak{G}_0) \times C^\infty(\mathcal{M}_1, \mathfrak{G}_1)$. The requirement

on $(\xi, \nu)$ to define a functor may be expressed in terms of point evaluations and linear maps, for instance, the compatibility with the source map is

$$s_{\mathfrak{G}}(\nu(x)) = \xi(s_{\mathcal{M}}(x)) \;\; \text{for all} \;\; x \in \mathcal{M}_1.$$

The same argument applies for the requirement on a functor to be $\mathcal{G}$-equivariant, and thus $\mathfrak{K}_0$ is a closed subalgebra in $\mathfrak{l}$.

In the same way, we may view $\mathfrak{K}_1$ as a closed subalgebra of $\mathfrak{K}_0 \times C^\infty(\mathcal{M}_0, \mathfrak{G}_1) \times \mathfrak{K}_0$, with $(F, \alpha, G) \in \mathrm{Mor}(\mathcal{C}^\infty(\mathcal{M}, \mathfrak{G}))$ if and only if $\alpha : F \Rightarrow G$ is a smooth natural equivalence. The structure maps are given by projections, embeddings and push-forwards by continuous linear mappings and thus all continuous algebra morphisms. ∎

The crucial tool in the description of the Lie group structure on $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G}_{\mathrm{Ad}})^{\mathcal{G}}$ shall be the exponential functions on $G$ and $H \rtimes G$ in the case that it provides charts for the Lie group structures (i.e., if $G$ and $H \rtimes G$ are *locally exponential*, cf. Appendix A).

**Lemma III.12.** *If $\mathcal{G}$ is a strict Lie 2-group, such that its group of objects and morphisms possess an exponential function, then these functions define a smooth functor*

$$\mathcal{E}xp : L(\mathcal{G}) \to \mathcal{G}.$$

**Proof.** For each homomorphism $\varphi : G_1 \to G_1$ between Lie groups with exponential function, the diagram

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\;\varphi\;} & G_2 \\[4pt]
\exp_1 \uparrow & & \exp_2 \uparrow \\[4pt]
\mathfrak{g}_1 & \xrightarrow{\;L(\varphi)\;} & \mathfrak{g}_2
\end{array}
\tag{17}
$$

eqn:exponentialMapCommutes

commutes. Since all requirements on $\mathcal{E}xp$ to define a functor can be phrased in such diagrams, the assertions follows. ∎

equivarianceOf2-Morphisms

**Remark III.13.** Let $\mathcal{M}$ be a strict $\mathcal{G}$-2-space. If the 2-morphism $\alpha : F \Rightarrow G$ between the morphisms $F, G$ of $\mathcal{G}$-2-spaces is viewed as a map $\alpha : \mathcal{M}_0 \to H \rtimes G$, then $\alpha = (\overline{\alpha}, F_0)$ for some $\overline{\alpha} : \mathcal{M}_0 \to H$ and $G$ satisfies satisfies $G_0 = (\beta \circ \overline{\alpha}) \cdot F_0$ and

$$G_1 = ((\overline{\alpha} \circ t_{\mathcal{M}}) \cdot \overline{F_1} \cdot (\alpha \circ s_{\mathcal{M}})^{-1}, G_0 \circ s_{\mathcal{M}}).$$

Thus $G$ is uniquely determined by $F$ and $\overline{\alpha}$. If $F$ and $\overline{\alpha}$ are is strictly equivariant, then so is $G$ since $\overline{\alpha} \in C^\infty(\mathcal{M}_0, H)^G$ by definition. ∎

For classical principal bundles, the compactness of the base manifold and the local exponentiality of the structure group ensure the existence of Lie group structures on gauge transformation groups (cf. [Woc07]). We shall follow similar ideas here and call a strict Lie 2-group *locally exponential* if its Lie groups of objects and morphisms are so.

**Theorem III.14.** *Assume that $M$ is compact, that $\mathcal{G}$ is locally exponential, and that the actions of $\mathcal{G}_1$ on $\mathcal{P}_1$ and of $\mathcal{G}_0$ on $\mathcal{P}_0$ are principal. Then $C^\infty(\mathcal{P}, \mathcal{G}_{\mathrm{Ad}})^{\mathcal{G}}$ is a locally exponential strict Lie 2-group with strict Lie 2-algebra $C^\infty(\mathcal{P}, L(\mathcal{G})_{\mathrm{ad}})^{\mathcal{G}}$.*

**Proof.** The proof works similar as in the case for classical principal bundles in [Woc07]. We set $\mathcal{K} := C^\infty(\mathcal{P}, \mathcal{G})^{\mathcal{G}}$, $\mathfrak{K} := C^\infty(\mathcal{P}, L(\mathcal{G}))^{\mathcal{G}}$ and denote by $\mathcal{K}_0$, $\mathfrak{K}_0$ and $\mathcal{K}_1$, $\mathfrak{K}_1$ the corresponding spaces of objects and morphisms. Then we have

$$\mathcal{K}_0 = \{(\gamma, \eta) \in C^\infty(\mathcal{P}_0, G)^G \times C^\infty(\mathcal{P}_1, H \rtimes G)^{H \rtimes G} : \eta \circ i_{\mathcal{P}} = i_{\mathcal{G}} \circ \gamma$$
$$\gamma \circ s_{\mathcal{P}} = s_{\mathcal{G}} \circ \eta, \ \gamma \circ t_{\mathcal{P}} = t_{\mathcal{G}} \circ \eta, \ \eta \circ c_{\mathcal{P}} = c_{\mathcal{G}} \circ (\eta_s \times_t \eta)\}.$$

Since the conditions on $(\gamma, \eta)$ to be in $\mathcal{K}_0$ can all be phrased in terms of evaluation maps on $\mathcal{P}_0$ and $\mathcal{P}_1$, it follows that $\mathcal{K}_0$ is a closed subgroup of $L := C^\infty(\mathcal{P}_0, G)^G \times C^\infty(\mathcal{P}_1, H \rtimes G)^{H \rtimes G}$, endowed with the $C^\infty$-topology. Similarly, we obtain $\mathfrak{K}_0$ as a closed subalgebra of $\mathfrak{l} := C^\infty(\mathcal{P}_0, \mathfrak{g})^G \times C^\infty(\mathcal{P}_1, \mathfrak{h} \rtimes \mathfrak{g})^{H \rtimes G}$.

Now $L$ is a locally convex, locally exponential Lie group, modelled on $\mathfrak{l}$ (cf. [Woc07, Thm. 1.11]), because the actions of $G$ on $\mathcal{P}_0$ and of $H \rtimes G$ on $\mathcal{P}_1$ are free and locally trivial (cf. Remark II.26). The exponential function for this Lie group is then given by

$$\mathfrak{l} \ni (\xi, \nu) \mapsto (\exp_G \circ \xi, \exp_{H \rtimes G} \circ \nu) \in L,$$

which restricts to a diffeomorphism on some zero neighbourhood in $\mathfrak{l}$. Since this is the same as the composition of the exponential functor $\mathcal{E}xp$ with $(\xi, \nu)$, this exponential function restricts to a map from $\mathfrak{K}_0$ to $\mathcal{K}_0$. It follows from the construction of the Lie group structure on $L$ that this map restricts to a bijective map of an open zero neighbourhood in $\mathfrak{K}_0$ to an open unit neighbourhood in $\mathcal{K}_0$. The compatibility with the structure maps of $\mathcal{P}$, $L(\mathcal{G})$ and $\mathcal{G}$ can be checked by repeated use of (17) in local coordinates, for all structure maps of $L(\mathcal{G})$ and $\mathcal{G}$ are (by the construction of $L(\mathcal{G})$) given by morphisms of Lie algebras and Lie groups, commuting with the respectively exponential functions. For instance

$$\exp_G \circ \xi \circ s_{\mathcal{P}} = s_{\mathcal{G}} \circ \exp_{H \rtimes G} \circ \nu \quad \Leftrightarrow \quad \xi \circ s_{\mathcal{P}} = s_{L(\mathcal{G})} \circ \nu.$$

if $\xi$ and $\nu$ have representatives in local coordinates, which take values in open zero neighbourhoods of $\mathfrak{g}$ and $\mathfrak{h} \rtimes \mathfrak{g}$, on which the exponential functions restricts respectively to a diffeomorphism. Thus $\mathcal{K}_0$ is a closed Lie subgroup of $L$.

In the same manner, one constructs $\mathcal{K}_1$ as a closed subgroup of $\mathcal{K}_0 \times C^\infty(\mathcal{P}_0, H \rtimes G)^G \times \mathcal{K}_0$ with $(F, \alpha, G) \in \mathcal{K}_1$ if and only if $\alpha : F \Rightarrow G$ is a smooth natural equivalence (cf. Remark III.13 and [NW09, Th. A.1], [Woc07, Thm. 1.11] for the Lie group structure on $C^\infty(\mathcal{P}_0, H \rtimes G)^G$). The exponential functions on $\mathcal{K}_0$ and $C^\infty(\mathcal{P}_0, H \rtimes G)^{H \rtimes G}$ induce an exponential function on $\mathcal{K}_1$ and as before, $\mathcal{K}_1$ is a closed Lie subgroup. The structure maps are given by projections, embeddings and push-forwards by Lie group morphisms and thus they all are morphisms of locally convex Lie groups. ∎

**Corollary III.15.** *If we endow the 2-group $C^\infty(\mathcal{P}, \mathcal{G}_{\mathrm{Ad}})^{\mathcal{G}}$ with the Lie 2-group structure from the previous theorem, then the natural action turns $\mathcal{P}$ into a smooth $C^\infty(\mathcal{P}, \mathcal{G}_{\mathrm{Ad}})^{\mathcal{G}}$-2-space.*

**Proof.** This follows from the fact that evaluation maps are smooth in the $C^\infty$-topology. ∎

**Remark III.16.** Taking Remark III.13 into account, one obtains that $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G}_{\mathrm{Ad}})^\mathcal{G}$ is associated to the smooth crossed module $(\alpha_*, \beta_*, C^\infty(\mathcal{P}_0, H)^G, \mathcal{M}or(\mathcal{P}, \mathcal{G})^\mathcal{G})$ with

$$\beta_*(\overline{\alpha}) = (\beta \circ \overline{\alpha}, (\overline{\alpha} \circ t) \cdot (\overline{\alpha} \circ s)^{-1})$$
$$(\alpha_*(\gamma_0, \gamma_1).\overline{\alpha})(p_0) = \gamma_0(p_0).\overline{\alpha}(p_0).$$

∎

**Remark III.17.** In [Gom06], there are constructed central extensions of gauge groups of (higher) abelian gerbes by the use of the cup-product in smooth Deligne (hyper-)cohomology $H^{n+2}(M, \mathbb{Z}_D^\infty(n+2))$. There the term gauge transformation is used for $H^{n+1}(M, \mathbb{Z}_D^\infty(n+1))$. It would be very interesting to explore the connection to our approach in order to get more general central extensions, for $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G})^\mathcal{G}$ in the non-abelian case (cf. Example IV.3 and [NW09]).∎

# IV    Examples

In this section, we provide some classes of examples of principal 2-bundles. This first example is an analogous construction of the simply connected cover of a connected manifold $M$ as a $\pi_1(M)$-principal bundle. It constructs for a simply connected manifold $N$ a principal $B\pi_2(N)$-2-bundle, where $B\pi_2(N)$ is the (discrete) 2-group associated to the crossed module $\pi_2(N) \to \{*\}$. For brevity, we shall restrict to the case where the manifold $N$ actually is a Lie group $G$ (not necessarily finite-dimensional, so $\pi_2(G) \neq 0$ in general). It already appeared implicitly at many places in the literature (e.g. in [BM94] and [Igl95]) but, as far as the author knows, it has not been worked out in terms of principal 2-bundles. It has the correct universal property for calling it *the* 2-connected cover of $G$ (cf. [PW09]).

**Example IV.1.** Let $G$ be a 1-connected Lie group. For each $g \in G$ we choose a continuous path from $\gamma_g$ from $e$ to $g$, such that $\gamma_e \equiv e$ and $\gamma_g$ depends continuously on $g$ on some unit neighbourhood. Moreover, since $G$ is 1-connected, we find for each pair $(g, h) \in G^2$ a continuous map $\eta_{g,h} : \Delta^2 \to G$ with

$$\partial \eta_{g,h} = \gamma_g + g.\gamma_{g^{-1}h} - \gamma_h,$$

where the sum on the right is taken in the group of singular 1-chains of $G$. Again, we assume $\sigma_{e,e} \equiv e$ and that $\sigma_{g,h}$ depends continuously on $g, h$ on some unit neighbourhood $U$ of $G$. With these choices we now set

$$\eta_{g,h,k} : gV \cap hV \cap kV \to \pi_2(G), \quad x \mapsto \underbrace{[\sigma_{e,g,h} + \sigma_{e,h,k} - \sigma_{e,g,k} + g.\sigma_{e,g^{-1}h,g^{-1}k}]}_{\text{tetrahedron with vertices } e,g,h,k} +$$

$$\underbrace{[g.(\sigma_{e,g^{-1}h,g^{-1}k} + \sigma_{e,g^{-1}k,k^{-1}x} - \sigma_{e,g^{-1}h,g^{-1}x} + g^{-1}h.\sigma_{e,h^{-1}k,h^{-1}x})]}_{\text{tetrahedron with vertices } g,h,k,x},$$

32

where $V \subseteq U$ is an open symmetric unit neighbourhood with $V^2 \subseteq U$. If $gV \cap hV \cap kV \neq \emptyset$, then $g^{-1}h$, $g^{-1}k$ and $h^{-1}k$ are elements of $V^2 \subseteq U$. Since $\sigma$ is continuous on $U$ the value of $\eta_{ghk}$ does not depend on $x$, and $\eta_{ghk}$ is constant and in particular smooth. It is easily verified from the above presentation that $\eta_{ghk} + \eta_{gkl} = \eta_{ghl} + \eta_{hkl}$ (note that this follows from the above formula only because we added the second tetrahedron, which does not contribute to the class in $\pi_2(G)$). Since $(V_g)_{g \in G}$ with $V_g := gV$ covers $G$, $(V_g, *, \eta_{g,h,k})$ defines a Čech cocycle with values in $\pi_2(G)$ and thus a principal $B\pi_2(G)$-2-bundle, where $B\pi_2(G)$ denotes the 2-group associated to the crossed module $\pi_2(G) \to \{*\}$. It is fairly easy to check that different choices of the above data lead to cohomologous cocycles. ∎

The remainder of this section deals with (lifting) bundle gerbes.

**Example IV.2.** We briefly recall (abelian) bundle gerbes as introduced in [Mur96] (cf. [ACJ05], [SW08c]). The class of gerbes that connect most naturally with our principal 2-bundles are lifting gerbes, so we will restrict to this class (the general case can easily be adapted). Let $(\alpha, \beta, H, G)$ be a smooth crossed module and $\pi : P \to M$ be a principal $G$-bundle. Moreover, we assume that $A := \ker(\beta) \hookrightarrow H \twoheadrightarrow G$ is a central extension (i.e., we assume $\beta$ to be surjective, cf. [Nee07, Sect. 3]).

There is a canonical map $f : P \times_M P \to G$, determined by $p = p' \cdot f(p, p')$ and we consider the pull-back principal $A$-bundle $Q := f^*(H)$ over $P \times_M P$. The question that one is interested in is whether the $A$-action on $Q$ extends to an $H$-action, turning $Q$ into a principal $H$-bundle over $M$ (cf. [LGW08], [Nee06a]).

With $\mathcal{G} = (G, H \rtimes G)$, one can cook up a principal $\mathcal{G}$-2-bundle $\mathcal{P}$ as follows. We define objects and morphisms by

$$\mathrm{Ob}(\mathcal{P}) = P \quad \mathrm{Mor}(\mathcal{P}) = Q,$$

where we identify $Q$ with $\{((p, p'), h) \in P \times P \times G : p = p' \cdot \beta(h)\}$. Source and the target map are given by $s((p, p'), h) = p'$, $t((p, p'), h) = p$, and composition of morphisms is then defined by $((p'', p), h') \circ ((p, p'), h) = ((p'', p'), h \cdot h')$ (the order of $h$ and $h'$ is important for $((p'', p'), h \cdot h')$ to be in $Q$ again). In order to match this with the bundle gerbes defined in [Mur96], note that this may also be viewed as a $A$-bundle morphism

$$\pi_{13}^*(Q) \to \pi_{12}^*(Q) \times \pi_{23}^*(Q)/A,$$

where $A$ acts on the right hand side via the embedding $A \hookrightarrow A \times A$, $a \mapsto (a, a^{-1})$ and $\pi_{ij} : P \times_M P \times_M P \to P \times_M P$ are the various possible projections.

Fixing local trivialisations $\pi \times g_i : P|_{U_i} \to U_i \times G$ for an open cover $(U_i)_{i \in I}$ defines the functors $\Phi_i : \mathcal{P}|_{U_i} \to U_i \times \mathcal{G}$ on objects and on morphisms we set $\Phi_i((p, p'), h) = (\pi(p), (g_i(p').h, g_i(p')))$. The action of $\mathcal{G}$ on $\mathcal{P}$ is given by the given $G$-action on objects and by

$$(((p, p'), h'), (h, g)) \mapsto ((p \cdot \beta(h) \cdot g, p' \cdot g), g^{-1}.(h' \cdot h))$$

33

on morphisms.

Note that the band of this bundle is the trivial bundle over $M$, because $\beta$ is surjective. In particular, it has nothing to do with the apparent bundle $P$, which serves only as a meaningless intermediate space. The outer action $G/\beta(H) \to \mathrm{Out}(H)$ is trivial and lifting gerbes are classified by $H^2(M, A)$ (cf. Remark II.29). ∎

The following example illustrates the close interplay between groups of sections in Lie group bundles (cf. [NW09]) and gauge groups of principal bundles (cf. [Woc07]).

**Example IV.3.** Consider the crossed module $\mathcal{A}ut(H)$, given by the conjugation morphism $H \to \mathrm{Aut}(H)$ and the natural action of $\mathrm{Aut}(H)$ on $H$. Assuming that $H$ is finite-dimensional and $\pi_0(H)$ is finitely generated, $\mathrm{Aut}(H)$ becomes a Lie group, modelled on $\mathrm{der}(\mathfrak{h})$ (cf. [Bou89]). Moreover, we assume that we are given a smooth action $\lambda : G \to \mathrm{Aut}(H)$ for Lie group $G$, such that the action lifts to a homomorphism $\varphi : G \to H^\flat := H/Z(H)$ (this is the case, e.g., if $G$ is connected and $\mathfrak{h}$ is semi-simple, for then the derived action lifts by [Hel78, Prop. II.6.4]).

A Lie group bundle now arises from a principal $G$-bundle $P \to M$ by taking the associated bundle $P \times_G H$. The group of sections of this bundle is isomorphic to the equivariant mapping group $C^\infty(P, H)^G$. Considering the associated principal $H^\flat$-bundle $P^\flat = P \times_\varphi H^\flat$, one can ask whether this principal $H^\flat$-bundle lifts to a principal $H$-bundle $P^\sharp$. In this case, one has

$$C^\infty(P^\sharp, H)^H \cong C^\infty(P^\flat, H)^{H^\flat} \cong C^\infty(P, H)^G$$

(as one can see in local coordinates), and the group of sections is actually the gauge group of $P^\sharp$.

In general, we associate to $\lambda$ the pull-back central extension $\varphi^*(H) \to G$ and thus obtain a strict 2-group $\mathcal{G}$. Then we associate to $P^\flat$ the principal $\mathcal{G}$-2-bundle $\mathcal{P}$ of the lifting gerbe associated to the central extension $Z(H) \hookrightarrow H \twoheadrightarrow H^\flat$ and obtain $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G})^{\mathcal{G}}$ as its gauge 2-group. From Remark III.16 we see that $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G})^{\mathcal{G}}$ is associated to a crossed module $(\alpha_*, \beta_*, H_*, G_*)$ with $H_* = C^\infty(P^\flat, H)^{H^\flat} \cong C^\infty(P, H)^G$ and $G_* \leq C^\infty(P^\flat, H^\flat)^{H^\flat} \times C^\infty(Q, H \rtimes H^\flat)^{H \times H^\flat}$. From the compatibility with the structure maps of $\mathcal{P}$ is follows that

$$(\gamma_0, (\overline{\gamma}_1, \gamma_0 \circ s_{\mathcal{P}})) \in G_* \Leftrightarrow \overline{\gamma}_1 \in C^\infty(Q, Z(H))^{H^\flat}$$

(note that $\gamma_0 \circ s_{\mathcal{P}} = \gamma_0 \circ t_{\mathcal{P}}$, because $s_{\mathcal{P}}(p) = t_{\mathcal{P}}(p) \cdot h$ with $h \in Z(H)$) and thus $G_* \cong C^\infty(P^\flat, H^\flat)^{H^\flat} \times C^\infty(M, Z(H))$. Thus $\mathcal{C}^\infty(\mathcal{P}, \mathcal{G})^{\mathcal{G}}$ is in general associated to the crossed module

$$C^\infty(P^\flat, H)^{H^\flat} \to C^\infty(P^\flat, H^\flat)^{H^\flat} \times C^\infty(M, Z(H)), \quad \gamma \mapsto (q \circ \gamma, e_H).$$

with the obvious point-wise action. ∎

**Example IV.4.** An instance of the previous example is given by considering the extension $\mathrm{Gau}(P) \hookrightarrow \mathrm{Aut}(P) \twoheadrightarrow \mathrm{Diff}(M)_P$ (cf. [Woc07]) for a finite-dimensional principal $K$-bundle $P \to M$, defining a crossed module by the conjugation action of $\mathrm{Aut}(P)$. If $P = M \times K$ is trivial, then $H := \mathrm{Gau}(P) \cong C^\infty(M,K)$ and $\mathrm{Aut}(P) \cong H \rtimes \mathrm{Diff}(M)$ and if $K$ is compact and simple, then $\mathrm{Aut}(P)$ is an open subgroup of $\mathrm{Aut}(H)$ (cf. [Gün09]). Then a given action $G \to \mathrm{Aut}(H)$ lifts to $H^\flat$, for instance, if $G$ is connected and the induced action $\mathfrak{g} \to \mathcal{V}(M)$ on the base space is trivial. ∎

# A    Appendix: Differential calculus on locally convex spaces

We provide some background material on spaces of mappings and their Lie group structure in this appendix.

**Definition A.1.** Let $X$ and $Y$ be a locally convex vector spaces and $U \subseteq X$ be open. Then $f : U \to Y$ is *differentiable* or $C^1$ if it is continuous, for each $v \in X$ the differential quotient

$$df(x).v := \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

exists and if the map $df : U \times X \to Y$ is continuous. If $n > 1$ we inductively define $f$ to be $C^n$ if it is $C^1$ and $df$ is $C^{n-1}$ and to be $C^\infty$ or *smooth* if it is $C^n$. We say that $f$ is $C^\infty$ or *smooth* if $f$ is $C^n$ for all $n \in \mathbb{N}_0$. We denote the corresponding spaces of maps by $C^n(U,Y)$ and $C^\infty(U,Y)$.

A (locally convex) *Lie group* is a group which is a smooth manifold modelled on a locally convex space such that the group operations are smooth. ∎

**Proposition A.2.** *Let $G$ be a group with a locally convex manifold structure on some subset $U \subseteq G$ with $e \in U$. Furthermore, assume that there exists $V \subseteq U$ open such that $e \in V$, $VV \subseteq U$, $V = V^{-1}$ and*

*i) $V \times V \to U$, $(g,h) \mapsto gh$ is smooth,*

*ii) $V \to V$, $g \mapsto g^{-1}$ is smooth,*

*iii) for all $g \in G$, there exists an open unit neighbourhood $W \subseteq U$ such that $g^{-1}Wg \subseteq U$ and the map $W \to U$, $h \mapsto g^{-1}hg$ is smooth.*

*Then there exists a unique locally convex manifold structure on $G$ which turns $G$ into a Lie group, such that $V$ is an open submanifold of $G$.*

**Definition A.3.** Let $G$ be a locally convex Lie group. The group $G$ is said to have an *exponential function* if for each $x \in \mathfrak{g}$ the initial value problem

$$\gamma(0) = e, \quad \gamma'(t) = T\lambda_{\gamma(t)}(e).x$$

35

has a solution $\gamma_x \in C^\infty(\mathbb{R}, G)$ and the function

$$\exp_G : \mathfrak{g} \to G, \quad x \mapsto \gamma_x(1)$$

is smooth. Furthermore, if there exists a zero neighbourhood $W \subseteq \mathfrak{g}$ such that $\exp_G|_W$ is a diffeomorphism onto some open unit neighbourhood of $G$, then $G$ is said to be *locally exponential*. ∎

**Lemma A.4.** *If $G$ and $G'$ are locally convex Lie groups with exponential function, then for each morphism $\alpha : G \to G'$ of Lie groups and the induced morphism $d\alpha(e) : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras, the diagram*

$$
\begin{array}{ccc}
G & \xrightarrow{\ \alpha\ } & G' \\[4pt]
\Big\uparrow{\scriptstyle \exp_G} & & \Big\uparrow{\scriptstyle \exp_{G'}} \\[4pt]
\mathfrak{g} & \xrightarrow{\ d\alpha(e)\ } & \mathfrak{g}'
\end{array}
$$

*commutes.*

**Remark A.5.** The Fundamental Theorem of Calculus for locally convex spaces (cf. [Glö02a, Th. 1.5]) yields that a locally convex Lie group $G$ can have at most one exponential function (cf. [Nee06b, Lem. II.3.5]).

Typical examples of locally exponential Lie groups are Banach-Lie groups (by the existence of solutions of differential equations and the inverse mapping theorem, cf. [Lan99]) and groups of smooth and continuous mappings from compact manifolds into locally exponential groups ([Glö02b, Sect. 3.2], [Woc06]). However, diffeomorphism groups of compact manifolds are never locally exponential (cf. [Nee06b, Ex. II.5.13]) and direct limit Lie groups not always (cf. [Glö05, Rem. 4.7]). For a detailed treatment of locally exponential Lie groups and their structure theory we refer to [Nee06b, Sect. IV]. ∎

**Remark A.6.** The most interesting examples of infinite-dimensional Lie groups for this article shall be groups of smooth mappings $C^\infty(M, G)$ from a compact manifold $M$ (possibly with boundary) to an arbitrary Lie group $G$. These groups possess natural Lie group structures if one endows them with the initial topology with respect to the embedding

$$C^\infty(M, G) \hookrightarrow \prod_{k \in \mathbb{N}_0} C(T^k M, T^k G)_c, \quad \gamma \mapsto (T^k \gamma)_{k \in \mathbb{N}_0}.$$

The Lie algebra is $C^\infty(M, \mathfrak{g})$ (with the above topology), where $\mathfrak{g}$ is the Lie algebra of $G$ (all spaces are endowed with point-wise operations). Details can be found in [Glö02b, Sect. 3.2] and [Woc06, Sect. 4]. ∎

## Acknowledgements

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